Continuous Random Variables p150

The type of random variables that can take any value in an interval is called continuous.

Defn 4.1 (Distribution function) p151
Let $Y$ denote any r.v. The distribution function of $Y$, denoted by $F(y)$, is given by

$$F(y) = P(Y \leq y) \quad \text{for} \quad -\infty < y < \infty.$$ 

Examples
Find the d.f of the following distributions

1) Uniform\{1,2,\ldots,6\}
2) Geo($p$)
**Theorem 4.1** p153
(Properties of a distribution function(d.f.))
If \( F(y) \) is a d.f., then
1) \( F(-\infty) = \lim_{y \to -\infty} F(y) = 0. \)
2) \( F(\infty) = \lim_{y \to \infty} F(y) = 1. \)
3) \( F(y) \) is a nondecreasing function of \( y \). (i.e \( y_1 < y_2 \Rightarrow F(y_1) \leq F(y_2) \))

**Defn 4.2** (p154)
Let \( Y \) be a r.v. with d.f \( F(y) \). \( Y \) is said to be continuous if the d.f. \( F(y) \) is continuous for \(-\infty < y < \infty\).

**Defn 3** (p154)
Let \( F(y) \) be the d.f. for a continuous r.v. \( Y \). Then \( f(y) \), given by
\[
f(y) = \frac{dF(y)}{dy} = F'(y)
\]
whenever the derivative exists, is called the probability density function (pdf) of the r.v. \( Y \).

Note. It follows from this definition that \( F(y) \) can be written as \( F(y) = \int_{-\infty}^{y} f(t) dt \).

**Theorem 4.2** p155
If \( f(y) \) is a density function, then
1. \( f(y) \geq 0 \) for any value of \( y \).
2. \( \int_{-\infty}^{\infty} f(y) dy = 1. \)

**Theorem 4.3** p157
If r.v \( Y \) has density function \( f(y) \) and \( a \leq b \), then the probability that \( Y \) falls in the interval \([a, b]\) is
\[
P(a \leq Y \leq b) = \int_{a}^{b} f(y) dy.
\]
Example 4.4 p159
Suppose that $Y$ has density function

$$f(y) = \begin{cases} ky(1-y), & 0 \leq y \leq 1 \\ 0, & \text{elsewhere}. \end{cases}$$

a) Find the value of $k$ that makes $f(y)$ is a p.d.f.

b) Find $P(0.4 \leq Y \leq 1)$

c) Find $P(0.4 \leq Y < 1)$

d) Find $P(Y \leq 0.4 \mid Y \leq 0.8)$.

e) d) Find $P(Y < 0.4 \mid Y < 0.8)$.

Ex 4.12 p161
Let $Y$ have the density function given by

$$f(y) = \begin{cases} 0.2 & -1 < y \leq 0 \\ 0.2 + cy & 0 < y \leq 1 \\ 0 & \text{elsewhere}. \end{cases}$$

a) Find $c$.

b) Find $F(y)$
c) Graph \( f(y) \) and \( F(y) \)
d) Find \( F(-1), F(0) \) and \( F(1) \).

e) Find \( P(0 \leq Y < 0.5) \)

f) \( P(Y > 0.5 \mid Y > 0.1) \)

Ex 4.5-4.13 p159
Expected Values for continuous Random Variables (4.3) p162

**Defn 4.4 p162**
The expected value of a continuous r. v. $Y$ is

$$E(Y) = \int_{-\infty}^{\infty} yf(y)dy$$

provided that the integral exists.

**Defn 4.5 p163**
Let $g(Y)$ be a function of $Y$; then the expected value of $g(Y)$ is given by

$$E[g(Y)] = \int_{-\infty}^{\infty} g(y)f(y)dy$$

provided that the integral exists.

**Theorem 4.5 p163**
Let $c$ be a constant, and $g(Y), g_1(Y), g_2(Y), \ldots, g_k(Y)$ be functions of a continuous r.v. $Y$. Then the following results hold.

1) $E(c) = c$
2) $E[cg(Y)] = cE[g(Y)]$
3) $E[g_1(Y) + g_2(Y) + \ldots + g_k(Y)] = E[g_1(Y)] + E[g_2(Y)] + \ldots + E[g_k(Y)]$

**Ex 4.14 p164**
If $Y$ has density function

$$f(y) = \begin{cases} \left(\frac{1}{2}\right)(2-y), & 0 \leq y \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

find the mean and variance of $Y$.

**Ex 4.19 p164**

**Ex 4.22 p165, 4.25 p165,**
**Uniform distribution p166**

**Defn 4.5 p166**
If \( \theta_1 < \theta_2 \), a r.v. \( Y \) is said to have a continuous uniform probability distribution on interval \((\theta_1, \theta_2)\) iff the density function of \( Y \) is

\[
f(y) = \begin{cases} 
\frac{1}{\theta_2 - \theta_1}, & \theta_1 \leq y \leq \theta_2 \\
0, & \text{elsewhere}
\end{cases}
\]

- Mean and the Variance of the \( \text{Unif}(\theta_1, \theta_2) \) distribution.

**Ex 4.32 p168**
The change in depth of a river from one day to the next (in feet) at a specific location, is a r.v. \( Y \) with the following p.d.f.

\[
f(y) = \begin{cases} 
k, & -2 \leq y \leq 2 \\
0, & \text{elsewhere.}
\end{cases}
\]

a) Determine the value of \( k \).
b) Obtain the distribution function of \( Y \).
The normal probability distribution 4.5, p170

- p.d.f p170
- mean and std dev p170
- std normal distribution
- reading tables

Ex 4.46 p173
Ex 4.47 p173
Ex 4.50 p173

Ex; 4.49 p173, 4.50, 4.60 p174
The Gamma Probability distribution (4.6) p.175

Defn 4.8 p176
A r. v. $Y$ is said to have a gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ iff the density function of $Y$ is given by

$$f(y) = \begin{cases} \frac{y^{\alpha-1}e^{-y/\beta}}{\beta^{\alpha}\Gamma(\alpha)} & 0 \leq y < \infty \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y} dy.$$

Some useful properties of the gamma function

1) $\Gamma(1) = 1$
2) For any $\alpha > 1$, $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$
3) For any positive integer $\alpha$, $\Gamma(\alpha) = (\alpha - 1)!$

Theorem 4.8 p177
$Y \sim \text{Gamma}(\alpha, \beta) \Rightarrow \mu = E(Y) = \alpha\beta$ and $\sigma^2 = V(Y) = \alpha\beta^2$.

Pf.
**Defn 4.9 p178 (chi-square distribution)**
Let $\nu$ be a positive integer. A r. v. $Y$ is said to have a chi-square distribution iff $Y \sim \text{Gamma}(\alpha = \nu / 2, \beta = 2)$.

**Theorem 4.9 p178**
If $Y$ is a chi-square r. v. with $\nu$ degrees of freedom, then $\mu = E(Y) = \nu$ and $\sigma^2 = V(Y) = 2\nu$.

**Defn 4.10 p178 (Exponential distribution)**
A r. v. $Y$ is said to have an exponential distribution with parameter $\beta > 0$ iff the density function of $Y$ is
$$f(y) = \begin{cases} 
\frac{1}{\beta} e^{-y/\beta}, & 0 \leq y < \infty \\
0, & \text{otherwise},
\end{cases}$$

**Theorem 4.10 p179**
If $Y$ is an exponential r. v. with parameter $\beta$, then $\mu = E(Y) = \beta$ and $\sigma^2 = V(Y) = \beta^2$.
Ex 4.10 179

$Y \sim \exp(\beta)$. Show that, if $a > 0$ and $b > 0$, 

$P(Y > a + b | Y > a) = P(Y > b)$.

This is called the memoryless property of the exponential distribution.

Ex 4.72 p180
Ex 4.82 p 181
Ex 4.84 p182
Ex 4.88 p180

Ex 4.67, 4.68, 4.71 p180, 4.76 4.77, 4.82,
The Beta distribution (4.7) p182

**Defn 4.11** p183

A r.v. \( Y \) is said to have a beta distribution with parameters \( \alpha > 0 \) and \( \beta > 0 \) iff the density function of \( Y \) is given by

\[
f'(y) = \begin{cases} 
y^{\alpha-1}(1-y)^{\beta-1} & 0 \leq y \leq 1 \\
0, & \text{otherwise,}
\end{cases}
\]

where

\[
B(\alpha, \beta) = \int_0^1 y^{\alpha-1} (1-y)^{\beta-1} dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.
\]

**Theorem 4.11** p184

If \( Y \) is a beta distributed r.v. with parameters \( \alpha > 0 \) and \( \beta > 0 \), then

\[
\mu = E(Y) = \frac{\alpha}{\alpha + \beta} \quad \text{and}
\]

\[
\sigma^2 = V(Y) = \frac{\alpha \beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)}.
\]

**Pf:**
Examples
Ex 4.91 p185
Ex 4.92 p185
Ex 4.96 p186
Ex 4.93, 4.94, 4.95, 4.97, 4.101

Moment-generating Functions of Continuous R.V.s
Ex 4.104, 4.105, 4.107, 4.108, 4.109, 4.111, 4.112, 4.133