Practice problems for test 1

1. Show that if the matrix $A \in \mathbb{R}^{n \times n}$ is real and $A^T = -A$ then $I_n - A$ is nonsingular and the matrix $(I - A)^{-1}(I + A)$ is orthogonal.

2. A matrix is triangular if it is upper triangular or lower triangular. Show that a square triangular matrix that is orthogonal is diagonal.

3. Show that if $A \in \mathbb{R}^{n \times p}$ has rank $p$ then $||A(A^T A)^{-1}A^T|| = 1$.

4. Show that if $A \in \mathbb{R}^{n \times n}$ has $||A|| > 1$ then $k(A) \geq 1$ where $k(A)$ is the condition number $||A|| \cdot ||A^{-1}||$.

5. Show that if $A \in \mathbb{R}^{n \times p}$ then the matrix $H = I - A(A^T A)^{-1}A^T$ has the property that $H^T = H$ and $H \cdot H = H$.

6. Let
   \[
   A = \begin{pmatrix}
   a_{11} & a_{12} \\
   a_{21} & a_{22}
   \end{pmatrix}
   \]
   What are the entries in the Givens rotation matrix that converts $a_{21}$ to zero?

7. (Forward substitution) Consider a matrix $L \in \mathbb{R}^{n \times n}$ that is lower triangular. Write the solution of the linear system $Lx = c$ where $x, c \in \mathbb{R}^n$ and $c$ is known.

8. Show that if $A \in \mathbb{R}^{n \times n}$ is symmetric then the eigenvalues of $A^T A$ are the squared eigenvalues of $A$.

9. Consider a given vector $x \in \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)^T$ and define $\tau = (0,0, \ldots, 0, \tau_{k+1}, \tau_{k+2}, \ldots, \tau_n)^T$ (the first $k$ entries are all zero) with $\tau_i = x_i/x_k$ for all $i = k + 1, k + 2, \ldots, n$. Define $M_k = I_n - \tau e_k^T$ where $e_k \in \mathbb{R}^n$ is the column vector which is all zero except for the $k$-th position which is one.

   Show that $M_k x = (x_1, x_2, \ldots, x_k, 0, 0, \ldots, 0)$, i.e. $M_k$ zeroes all entries in $x$ of order larger than $k$. 

1