Stochastic Process - Introduction

- Stochastic processes are processes that proceed randomly in time.
- Rather than consider fixed random variables X, Y, etc. or even sequences of i.i.d random variables, we consider sequences X_0, X_1, X_2, …. Where X_t represent some random quantity at time t.
- In general, the value X_t might depend on the quantity X_{t-1} at time t-1, or even the value X_s for other times $s < t$.
- Example: simple random walk .

Stochastic Process - Definition

- A stochastic process is a family of time indexed random variables X_t where t belongs to an index set. Formal notation, $\{X_t : t \in I\}$ where I is an index set that is a subset of R.

- Examples of index sets:
  1) I = (-∞, ∞) or I = [0, ∞]. In this case X_t is a continuous time stochastic process.
  2) I = {0, ±1, ±2, ….} or I = {0, 1, 2, …}. In this case X_t is a discrete time stochastic process.

- We use uppercase letter $\{X_t\}$ to describe the process. A time series, $\{x_t\}$ is a realization or sample function from a certain process.

- We use information from a time series to estimate parameters and properties of process $\{X_t\}$.

Probability Distribution of a Process

- For any stochastic process with index set *I*, its probability distribution function is uniquely determined by its finite dimensional distributions.

- The *k* dimensional distribution function of a process is defined by

$$F_{X_{t_1},\ldots,X_{t_k}}\left(x_1,\ldots,x_k\right) = P\left(X_{t_1} \le x_1,\ldots,X_{t_k} \le x_k\right)$$

  for any $t_1,\ldots,t_k \in I$ and any real numbers $x_1, \ldots, x_k$.

- The distribution function tells us everything we need to know about the process $\{X_t\}$.

Moments of Stochastic Process

- We can describe a stochastic process via its moments, i.e., $E(X_t), E(X_t^2), E(X_t \cdot X_s)$ etc. We often use the first two moments.

- The mean function of the process is $E(X_t) = \mu_t$.

- The variance function of the process is $Var(X_t) = \sigma_t^2$.

- The covariance function between X_t , X_s is

$$\text{Cov}(X_t, X_s) = E((X_t - \mu_t)(X_s - \mu_s))$$

- The correlation function between X_t , X_s is

$$\rho(X_t, X_s) = \frac{\text{Cov}(X_t, X_s)}{\sqrt{\sigma_t^2}\sqrt{\sigma_s^2}}$$

- These moments are often function of time.

Stationary Processes

- A process is said to be strictly stationary if $\left(X_{t_1},\ldots,X_{t_k}\right)$ has the same joint distribution as $\left(X_{t_1+\Delta},\ldots,X_{t_k+\Delta}\right)$. That is, if

$$F_{X_{t_1},\ldots,X_{t_k}}\left(x_1,\ldots,x_k\right)=F_{X_{t_1+\Delta},\ldots,X_{t_k+\Delta}}\left(x_1,\ldots,x_k\right)$$

- If $\{X_t\}$ is a strictly stationary process and $E\left(X_t^2\right)<\infty$ then, the mean function is a constant and the variance function is also a constant.

- Moreover, for a strictly stationary process with first two moments finite, the covariance function, and the correlation function depend only on the time difference s.

- A trivial example of a strictly stationary process is a sequence of i.i.d random variables.

Weak Stationarity

- Strict stationarity is too strong of a condition in practice. It is often difficult assumption to assess based on an observed time series $x_1,\ldots,x_k$.
- In time series analysis we often use a weaker sense of stationarity in terms of the moments of the process.
- A process is said to be *n*th-order weakly stationary if all its joint moments up to order *n* exists and are time invariant, i.e., independent of time origin.
- For example, a second-order weakly stationary process will have constant mean and variance, with the covariance and the correlation being functions of the time difference along.
- A strictly stationary process with the first two moments finite is also a second-ordered weakly stationary. But a strictly stationary process may not have finite moments and therefore may not be weakly stationary.

The Autocovariance and Autocorrelation Functions

- For a stationary process $\{X_t\}$, with constant mean μ and constant variance σ_2. The covariance between X_t and X_{t+s} is

$$\gamma(s) = \text{cov}(X_t, X_{t+s}) = E((X_t - \mu)(X_s - \mu))$$

- The correlation between X_t and X_{t+s} is

$$\rho(s) = \frac{\text{cov}(X_t, X_s)}{\sqrt{\text{var}(X_t)}\sqrt{\text{var}(X_s)}} = \frac{\gamma(s)}{\gamma(0)}$$

Where $\text{var}(X_t) = \text{var}(X_{t+s}) = \gamma(0)$.

- As functions of s, $\gamma(s)$ is called the autocovariance function and $\rho(s)$ is called the autocorrelation function (ATF). They represent the covariance and correlation between X_t and X_{t+s} from the same process, separated only by s time lags.

Properties of γ(*s*) and ρ(*s*)

- For a stationary process, the autocovariance function γ(*s*) and the autocorrelation function ρ(*s*) have the following properties:
  - $\gamma(0) = \mathrm{var}(X_t); \quad \rho(0) = 1.$
  - $-1 \le \rho(s) \le 1$.
  - $\gamma(s) = \gamma(-s)$ and $\rho(s) = \rho(-s)$.
  - The autocovariance function γ(*s*) and the autocorrelation function ρ(*s*) are positive semidefinite in the sense that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j\gamma(i-j) \ge 0 \qquad \text{and} \qquad \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j\rho(i-j) \ge 0$$

for any real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Correlogram

- A correlogram is a plot of the autocorrelation function $\rho(s)$ versus the lag s where $s = 0,1, \ldots.$

- Example…

Partial Autocorrelation Function

- Often we want to investigate the dependency / association between X_t and X_{t+k} adjusting for their dependency on $X_{t+1}, X_{t+2},\ldots, X_{t+k-1}$.
- The conditional correlation $\text{Corr}(X_t , X_{t+k} \mid X_{t+1}, X_{t+2},\ldots, X_{t+k-1})$ is usually referred to as the partial correlation in time series analysis.
- Partial autocorrelation is usually useful for identifying autoregressive models.

Gaussian process

- A stochastic process is said to be a normal or Gaussian process if its joint probability distribution is normal.
- A Gaussian process is strictly and weakly stationary because the normal distribution is uniquely characterized by its first two moments.
- The processes we will discuss are assumed to be Gaussian unless mentioned otherwise.
- Like other areas in statistics, most time series results are established for Gaussian processes.

White Noise Processes

- A process $\{X_t\}$ is called white noise process if it is a sequence of uncorrelated random variables from a fixed distribution with constant mean μ (usually assume to be 0) and constant variance σ^2.
- A white noise process is stationary with autocovariance and autocorrelation functions given by ….
- A white noise process is Gaussian if its joint distribution is normal.

Estimation of the mean

- Given a single realization $\{x_t\}$ of a stationary process $\{X_t\}$, a natural estimator of the mean $E(X_t) = \mu$ is the sample mean

$$\bar{x} = \frac{1}{n}\sum_{t=1}^{n} x_t$$

which is the time average of *n* observations.

- It can be shown that the sample mean is unbiased and consistent estimator for μ.

Sample Autocovariance Function

- Given a single realization $\{x_t\}$ of a stationary process $\{X_t\}$, the sample autocovariance function given by

$$\hat{\gamma}(k) = \frac{1}{n}\sum_{t=1}^{n=k}(x_t - \bar{x})(x_{t+k} - \bar{x})$$

is an estimate of the autocivariance function.

Sample Autocorrelation Function

- For a given time series $\{x_t\}$, the sample autocorrelation function is given by

$$\hat{\rho}(k) = \frac{\sum_{t=1}^{n-k}(x_t - \overline{x})(x_{t+k} - \overline{x})}{\sum_{t=1}^{n}(x_t - \overline{x})^2} = \frac{\hat{\gamma}(k)}{\hat{\gamma}(0)}.$$

- The sample autocorrelation function is non-negative definite.
- The sample autocovariance and autocorrelation functions have the same properties as the autocovariance and autocorrelation function of the entire process.

Example