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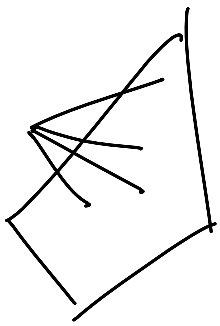
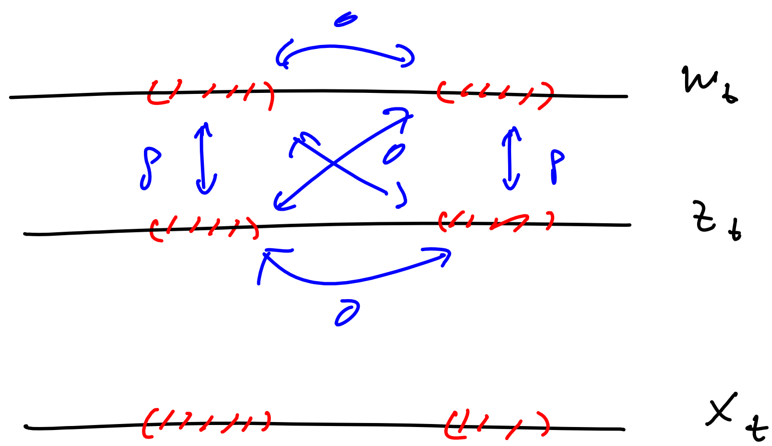
c) $X_t = a W_t + b Z_t$

B. motn   B. motn   corr ρ.



$W_t = B^1_t$

$Z_t = \rho B^1_t + \sqrt{1-\rho^2}\, B^2_t$



stationary increments.

$$X_t - X_s = a(W_t - W_s) + b(Z_t - Z_s)$$

$\overset{?}{\overset{d}{=}} X_{t-s}$?

$$\overset{d}{=} a W_{t-s} + b Z_{t-s}$$

$$= X_{t-s}$$

distribution property:

$X_t = aW_t + bZ_t$ is normal b.c.

$$\begin{pmatrix} W_t \\ Z_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} t & \rho t \\ \rho t & t \end{pmatrix} \right)$$

$\mathbb{E}[X_t] = 0$

$$\mathbb{V}[X_t] = a^2 \mathbb{V}[W_t] + b^2 \mathbb{V}[Z_t] + 2ab\, \mathbb{C}[W_t, Z_t]$$

$$= \underbrace{(a^2 + b^2 + 2ab\rho)}_{=: c^2} t$$

$Y_t = \frac{1}{c} X_t$ is a std. B. motion.

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$$X_t = W_t Z_t$$

- X_t has continuous paths
- $X_0 = 0$
- X_t ind. incr.
-

$$X_t - X_s = W_t Z_t - W_s Z_s \overset{?}{\underset{d}{=}} X_{t-s} \quad \text{no.}$$

$$(W_t - W_s)(Z_t - Z_s) = W_t Z_t - W_s Z_s$$

$$(-2W_s Z_s)$$

$$(-W_s Z_t - W_t Z_s)$$

$$\begin{aligned} W_t Z_t - W_s Z_s = &(W_t - W_s)(Z_t - Z_s) \\ &+ W_s Z_t + W_t Z_s \\ &- W_s Z_s - W_s Z_s \end{aligned}$$

$$\begin{aligned} &(W_t - W_s)(Z_t - Z_s) \\ &+ W_s (Z_t - Z_s) + Z_s (W_t - W_s) \end{aligned}$$

$$\mathbb{E}[X_t] = \mathbb{E}[W_t Z_t] = \rho t$$

$$\mathbb{E}[X_t^2] = \mathbb{E}[W_t^2 Z_t^2]$$

uncorr to W_t

$$= \mathbb{E}[W_t^2 (\rho W_t + \sqrt{1-\rho^2}\, B_t)^2]$$

$$= \mathbb{E}[\ \rho^2 W_t^4 + 2\rho\sqrt{1-\rho^2}\, W_t^3 B_t$$

$$+ (1-\rho^2) W_t^2 B_t^2\ ]$$

$$= \rho^2 \underbrace{\mathbb{E}[W_t^4]}_{\to 3t^2} + 2\rho\sqrt{1-\rho^2}\, \underbrace{\mathbb{E}[W_t^3]}_{\to 0} \underbrace{\mathbb{E}[B_t]}_{\to 0}$$

$$+ (1-\rho^2) \underbrace{\mathbb{E}[W_t^2]}_{\hookrightarrow t} \underbrace{\mathbb{E}[B_t^2]}_{\hookrightarrow t}$$

$$= (2\rho^2 + 1)\, t^2$$

$$\mathbb{E}[W_t^4] = \mathbb{E}[(\sqrt{t}\, N)^4] \qquad N \sim \mathcal{N}(0,1)$$

$$= t^2\ \mathbb{E}[N^4]$$

$$\mathbb{E}[\ e^{aN}] = e^{\frac{1}{2}a^2}, \quad \lim_{a \downarrow 0} \partial_a^4\, \mathbb{E}[e^{aN}] = \mathbb{E}[N^4]$$

$$\partial_a (e^{\frac{1}{2}a^2}) = a\, e^{\frac{1}{2}a^2}$$

$$\partial_a^2 (e^{\frac{1}{2}a^2}) = (1 + a^2)\, e^{\frac{1}{2}a^2}$$

$$\partial_a^3 (e^{\frac{1}{2}a^2}) = (2a + a(1+a^2))\, e^{\frac{1}{2}a^2}$$
$$= (3a + a^3)\, e^{\frac{1}{2}a^2}$$

$$\partial_a^4 (e^{\frac{1}{2}a^2}) = (3 + a^2 + a(3a + a^3))\, e^{\frac{1}{2}a^2}$$

$$\to 3.$$

$$\int_{-\infty}^{\infty} x^4 e^{-\frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}}$$

$$\partial_\alpha^2 \int_{-\infty}^{\infty} e^{-\frac{a}{2}x^2} \frac{dy}{\sqrt{2\pi}} = -\frac{1}{4}\int_{-\infty}^{\infty} x^4 e^{-\frac{a}{2}x^2} \frac{dy}{\sqrt{2\pi}}$$

$$X_{s+t} - X_s \overset{d}{\underset{?}{=}} X_t$$

$$\mathbb{E}[X_{s+t} - X_s]$$

$$= \mathbb{E}[W_{s+t} Z_{s+t} - W_s Z_s] = \rho(s+t) - \rho s$$

$$= \rho t$$

$$\mathbb{E}[(X_{s+t} - X_s)^2]$$

$$= \mathbb{E}[W_{s+t}^2 Z_{s+t}^2 + W_s^2 Z_s^2 - 2 W_{s+t} Z_{s+t} W_s Z_s]$$

$$= (2\rho^2 + 1)(s+t)^2 + (2\rho^2+1)s^2$$

$$\rightarrow Z_s + (Z_{s+t} - Z_s)$$

$$- 2\,\underbrace{\mathbb{E}[W_{s+t} Z_{s+t} W_s Z_s]}_{A}$$

$$\hookrightarrow W_s + (W_{s+t} - W_s)$$

$$A = \mathbb{E}[(W_s + (W_{s+t} - W_s))(Z_s + (Z_{s+t} - Z_s))$$

$W_s \, Z_s]$

$(2\rho^2+1)s^2$ $\qquad$ $\mathbb{E}[(W_{s+t}-W_s)]\,\mathbb{E}[W_s Z_s^2] \to 0$

$$= \mathbb{E}[W_s^2 Z_s^2] + \mathbb{E}[(W_{s+t}-W_s) W_s Z_s^2] + 0$$

$$+ \mathbb{E}[(W_{s+t}-W_s)(Z_{s+t}-Z_s) \, W_s Z_s]$$

$\hookrightarrow \rho t \cdot \rho s$

$$\mathbb{E}[W_s Z_s^2] = \mathbb{E}[(\rho Z_s + \sqrt{1-\rho^2}\, B_s) Z_s^2]$$

$$= \mathbb{E}[\rho Z_s^3 + \sqrt{1-\rho^2}\, B_s Z_s^2] = 0$$

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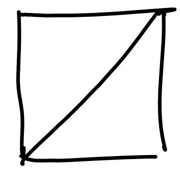
$$\mathbb{E}\left[\left(\int_0^t W_s Z_s \, ds\right)^2\right]$$

$$= \int_0^t \int_0^t \mathbb{E}[W_s Z_s W_u Z_u] \, ds \, du$$



$$\hookrightarrow Z_s + (Z_u - Z_s)$$

$$= 2 \int_0^t \int_0^u \mathbb{E}[W_s Z_s W_u Z_u] \, ds \, du$$

$$\hookrightarrow W_s + (W_u - W_s)$$

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$$Y_t = \int_0^t s^2 W_s^2 \, dW_s$$

$$\mathbb{E}[Y_t] = 0$$

$$\mathbb{E}[Y_t] \neq \int_0^t \mathbb{E}[s^2 W_s^2] \, dW_s$$

$$\mathbb{E}[Y_t] = \int_0^t \mathbb{E}[s^2 W_s^2 \, dW_s]$$

$$= 0$$

$$\mathbb{V}[Y_t] = \mathbb{E}[Y_t^2] = \mathbb{E}\left[\int_0^t (s^2 W_s^2)^2 \, ds\right]$$

$$\text{"} (dW_t)^2 \to dt \text{"}$$

$$Y_t = \int_0^t s^2 W_s^2 \, dW_s$$

$$g(w,t) = \frac{1}{3} t^2 w^3$$

$\frac{1}{2} \partial_{ww}$

$$d\, g(W_t,t) = (\partial_t + \mathcal{L})\, g \, dt + \partial_w g(W_t,t)\, dW_t$$

$$d\left(\frac{1}{3} t^2 W_t^3\right) = \left(\frac{2}{3} t W_t^3 + \frac{1}{2} 2 . t^2 . W_t\right) dt$$

$$+ t^2 W_t^2 \, dW_t$$

$$\frac{1}{3} t^2 W_t^3 - \frac{1}{3} 0^2 W_0^3$$

$$- \int_0^t \left(\frac{2}{3} s W_s^3 + s^2 W_s \right) ds = \int_0^t s^2 W_s^2 \, dW_s$$

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$$Y_t = \int_0^t \underline{s W_s} \, dZ_s$$

$$\mathbb{E}[Y_t] = 0$$

$$\mathbb{E}[Y_t^2] = \mathbb{E}\left[\int_0^t s^2 W_s^2 \, ds\right] = \ldots$$

$$g(t, w, z) = s w z$$

$$dg(t, W_t, Z_t) = (\partial_t + \mathcal{L}) g(t, W_t, Z_t) \, dt$$

$$+ \partial_w g \, dW_t + \underline{\partial_z g} \, dZ_t$$

$$\mathcal{L} = \tfrac{1}{2} \partial_{ww} + \tfrac{1}{2} \partial_{zz} + \underline{\rho} \, \partial_{wz}$$

$$d(t W_t Z_t) = (W_t Z_t + \rho t) \, dt$$

$$+ t Z_t \, dW_t + \underline{t W_t \, dZ_t}$$

$$\Rightarrow \int_0^t s W_s \, dZ_s = t W_t Z_t - \int_0^t (W_s Z_s + \rho s) \, ds$$

$$- \int_0^t s Z_s \, dW_s$$

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$$\frac{dS_t}{S_t} = \mu_t \, dt + \sigma_t \, dW_t^S$$

$$\frac{dU_t}{U_t} = \nu_t \, dt + \eta_t \, dW_t^u$$

$$g(s,u) = s^\alpha u^\beta \quad , \quad Y_t = g(S_t, U_t)$$

$$dY_t = (\partial_t + \mathcal{L}) g \, dt + \sigma_t S_t \cdot \partial_s g \, dW_t^S$$
$$+ \eta_t U_t \, \partial_u g \, dW_t^u$$

$$\mathcal{L} = \tfrac{1}{2} \cdot \sigma_t^2 s^2 \, \partial_{ss} + \tfrac{1}{2} \cdot \eta_t^2 u^2 \, \partial_{uu}$$
$$+ \sigma_t s \cdot \eta_t u \cdot \rho \cdot \partial_{su}$$
$$+ \mu_t s \, \partial_s + \nu_t u \, \partial_u$$

$$dY_t = \Big[\; 0 + \mu_t S_t \cdot \underbrace{S_t^{\alpha-1} \cdot \alpha \, U_t^\beta}_{\partial_s}$$
$$+ \nu_t U_t \cdot \underbrace{S_t^\alpha \, \beta \, U_t^{\beta-1}}_{\partial_u}$$
$$+ \tfrac{1}{2} \cdot \sigma_t^2 S_t^2 \cdot \underbrace{S_t^{\alpha-2} \alpha(\alpha-1) U_t^\beta}_{\partial_{ss}}$$

$$+ \quad \sigma_t \eta_t \rho \, U_t S_t \cdot \underbrace{S_t^{\alpha-1} \alpha \, U_t^{\beta-1} \beta}_{\partial_{su}}$$

$$+ \frac{1}{2} \eta_t^2 U_t^2 \cdot \underbrace{S_t^{\alpha} U_t^{\beta-2} \beta(\beta-1)}_{\partial_{uu}} \Big] dt$$

$$+ \quad \sigma_t S_t \cdot S_t^{\alpha-1} \alpha \, U_t^{\beta} \, dW_t^s$$

$$+ \quad \eta_t U_t \, S_t^{\alpha} U_t^{\beta-1} \beta \, dW_t^u$$

$$= S_t^{\alpha} U_t^{\beta} \Big[\underbrace{\alpha \mu_t + \beta \nu_t + \frac{1}{2} \sigma_t^2 \alpha(\alpha-1) + \frac{1}{2} \eta_t^2 \beta(\beta-1) + \sigma_t \eta_t \rho \alpha \beta}_{a_t} \Big] dt$$

$$+ S_t^{\alpha} U_t^{\beta} \left(\sigma_t \alpha \, dW_t^s + \eta_t \beta \, dW_t^u \right)$$

$$\frac{d(S_t^{\alpha} U_t^{\beta})}{S_t^{\alpha} U_t^{\beta}} = a_t \, dt + \underbrace{\alpha \sigma_t dW_t^s + \beta \eta_t dW_t^u}_{dX_t}$$

$$X_t = \int_0^t \alpha \sigma_u dW_u^s + \int_0^t \beta \eta_u dW_t^u$$

$$\overset{d}{=} \int_0^t \gamma_u \, dB_u$$