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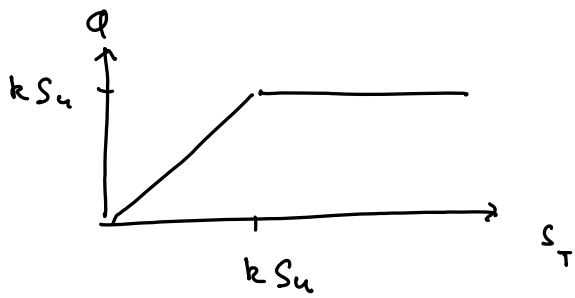
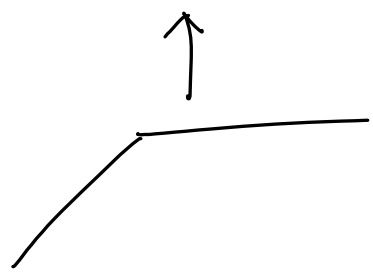
Δ for i) $S_T \, \mathbb{1}_{S_T > K}$

ii) $\min(S_T, k S_u)$, $u < T$

$$V_t = \begin{cases} S_t \, h(t), & t < u \\ g(S_t, S_u, t), & t \geq u \end{cases}$$





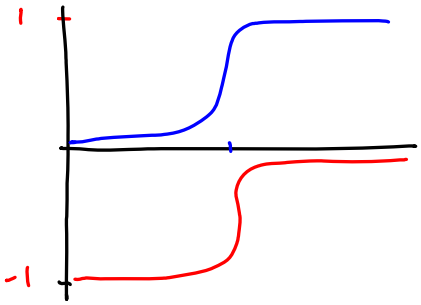
$$Q = k S_u - (k S_u - S_T)_+ \qquad @ \, T$$

$t > u$

$$V_t = k S_u e^{-r(T-t)} - \left[k S_u e^{-r(T-t)} \Phi(-d_-) - S_t \Phi(-d_+) \right]$$

$$d_\pm = \frac{\ln(S_t / k S_u) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}}$$

$$\Delta_t = + \Phi(-d_+)$$



$$\Delta_t^{call} = \Phi(d_+)$$

$$\Delta_t^{put} = \Phi(d_+) - 1$$

$$= -\Phi(-d_+)$$

$t \le u$,

$$V_t = e^{-rT}\,\mathbb{E}^{Q}\big[\,\mathbb{E}^{Q}[\,Q \mid \mathcal{F}_u\,] \mid \mathcal{F}_t\big]$$

$$= e^{-ru}\,\mathbb{E}^{Q}\big[\,\underbrace{e^{-r(T-u)}\,\mathbb{E}^{Q}[\,Q \mid \mathcal{F}_u\,]}_{V_u} \mid \mathcal{F}_t\big]$$

$$V_u = S_u\Big[\,k\,e^{-r(T-u)} - \big[k\,\Phi(-d_-) - \Phi(-d_+)\big]\Big]$$

α

$$d_\pm = \frac{\ln(1/k) + (r \pm \frac{1}{2}\sigma^2)(T-u)}{\sigma\sqrt{T-u}}$$

$$\Rightarrow \quad V_t = S_t\,\alpha$$

$$\Delta_t = \alpha$$

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$$V_t^{call} = S_t \, \Phi(d_+) - K e^{-r(T-t)} \, \Phi(d_-)$$

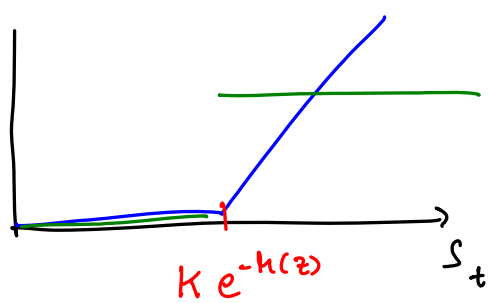
$$\Delta_t = \Phi(d_+)$$

$$V_t^{call} = e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}}\left[(S_T - K)_+ \mid \mathcal{F}_t \right]$$

$$= e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}}\left[\left(S_t \, e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\, Z} - K \right)_+ \right]$$

$$Z \underset{\mathbb{Q}}{\sim} N(0,1)$$

$$\frac{\partial V_t}{\partial S_t} = e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}}\left[\frac{\partial}{\partial S_t} \left(S_t \, e^{h(z)} - K \right)_+ \right]$$



$$= \frac{e^{-r(T-t)}}{S_t} \, \mathbb{E}^{\mathbb{Q}}\left[S_t e^{h(z)} \, \mathbb{1}_{S_t e^{h(z)} > K} \right]$$

$$= \frac{V_t}{S_t}$$

$$V_t = e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}}\left[S_T \, \mathbb{1}_{S_T > K} \mid \mathcal{F}_t \right]$$

$$\Delta_t = \frac{V_t}{S_t} = \mathbb{E}^{\mathbb{Q}^S}\left[\frac{\cancel{S_T}\, \mathbb{1}_{S_T > K}}{\cancel{S_T}} \,\middle|\, \mathcal{F}_t \right]$$

$$= \mathbb{Q}^S(S_T > K) = \Phi(d_+)$$

$$S_T \overset{d}{=} S_t \exp\left\{ \left(r + \tfrac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}\, Z \right\}$$

$$Z \underset{\mathbb{Q}_S}{\sim} \mathcal{N}(0,1)$$

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$$V_t = S\,\Phi(d_+) - K e^{-r(T-t)}\,\Phi(d_-)$$

$$\frac{\partial V_t}{\partial S} = \Phi(d_+) + \left(\frac{S}{K}\Phi'(d_+)\sqrt{2\pi} - e^{-r(T-t)}\sqrt{2\pi}\,\Phi'(d_-)\right)\frac{d'_\pm K}{\sqrt{2\pi}}$$

$$d'_\pm = \frac{1}{S\,\sigma\sqrt{T}}$$

$$\Phi'(x) = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$$

$$\frac{S}{K}\,e^{-\frac{1}{2}d_+^2} \qquad\qquad e^{-rT}\,e^{-\frac{1}{2}d_-^2}$$

$$\ln\left(\frac{S}{K}\right) - \frac{1}{2}d_+^2 \qquad\qquad -rT - \frac{1}{2}d_-^2$$

$$= \ln\left(\frac{S}{K}\right) - \frac{1}{2}\left(\frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)^2$$

$$= \ln\left(\frac{S}{K}\right) - \frac{1}{2}\cancel{\frac{(\ln(S/K))^2}{\sigma^2 T}} - \frac{1}{2}\frac{\cancel{(r + \frac{1}{2}\sigma^2)^2}\,T^2}{\sigma^2 T} \qquad \sigma^2 r$$

$$- \frac{\ln(S/K)\,(\cancel{r} + \frac{1}{2}\sigma^2 T)}{\sigma^2 T} \qquad - \sigma^2 r$$

$$-rT - \frac{1}{2}\frac{\left(\ln(S/K)\right)^2}{\sigma^2 T} - \frac{1}{2}\frac{\left(r - \frac{1}{2}\sigma^2\right)^2 T^2}{\sigma^2 T}$$

$$- \frac{\ln(S/K)\left(\cancel{r} - \frac{1}{2}\sigma^2 T\right)}{\sigma^2 T}$$

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Solutions of a PDE of the following kind:

$$\begin{cases} \partial_t g + a(x,t)\,\partial_x g + \frac{1}{2} b^2(x,t)\,\partial_{xx} g = c(x,t)\, g \\ g(x,T) = \Phi(x) \end{cases}$$

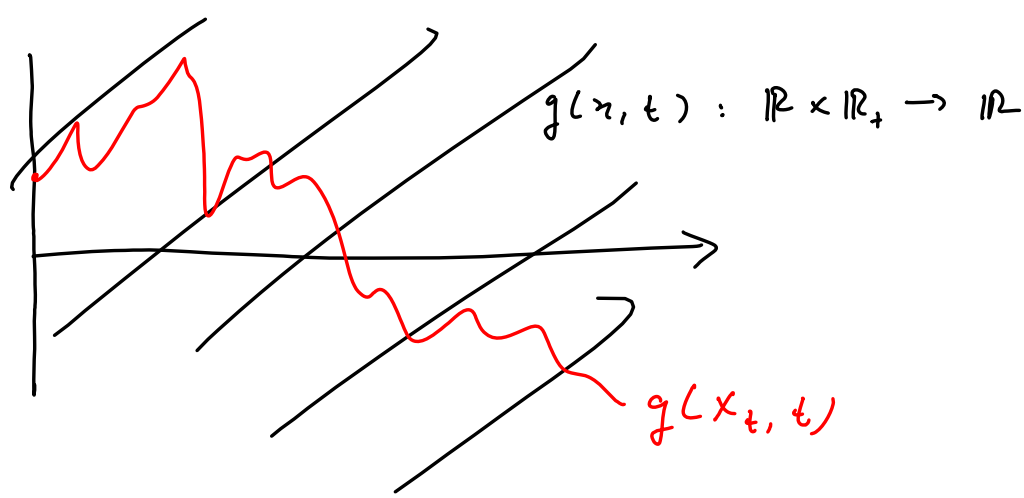
can be written as follows

$$g(x,t) = \mathbb{E}^{\mathbb{M}}\left[e^{-\int_t^T c(X_s,s)\,ds}\, \Phi(X_T) \mid \mathcal{F}_t \right]$$

where,

$$\begin{cases} dX_t = a(X_t,t)\,dt + b(X_t,t)\,dW_t \\ X_t = x \end{cases}$$

W_t is a $\mathbb{M}$-Brownian mtn, $\mathcal{F}_t$ is the natural filtration of W_t



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$$\begin{cases} \partial_t g + \partial_x g + \frac{1}{2}\partial_{xx} g = 0 \\ g(x,T) = x^2 \end{cases}$$

$$g(x,t) = \mathbb{E}^{\mathbb{M}}[X_T^2 \mid \mathcal{F}_t]$$

$$\begin{cases} dX_t = 1 \cdot dt + 1 \cdot dW_t \\ X_t = x \end{cases}$$

$$X_T - \underset{\overset{\shortparallel}{x}}{X_t} = (T-t) + (W_T - W_t)$$

$$X_T = (x + (T-t)) + (W_T - W_t)$$

$$X_T^2 = (x + (T-t))^2 + (W_T - W_t)^2 + 2(W_T - W_t)(x + (T-t))$$

$$g(x,t) = (x + (T-t))^2 + (T-t) + 0$$

$$= x^2 + (2x+1)(T-t) + (T-t)^2$$

$$g(x,T) = \mathbb{1}_{x>1}$$

$$g(x,t) = \mathbb{E}^{\mathbb{M}}[\mathbb{1}_{X_T>1} \mid \mathcal{F}_t]$$

$$= \mathbb{M}(X_T > 1)$$

$$= \mathbb{M}\left(x + (T-t) + W_T - W_t > 1 \right)$$

$$= \mathbb{M}\left(\frac{W_T - W_t}{\sqrt{T-t}} > \frac{1-(x+(T-t))}{\sqrt{T-t}} \right)$$

$$= \Phi\left(\frac{x+(T-t) - 1}{\sqrt{T-t}} \right)$$

$$\partial_t g = \Phi'(-)\, \partial_t \left(\frac{x-1}{\sqrt{T-t}} + \sqrt{T-t} \right)$$

$$= \Phi'(-) \left(+\frac{1}{2} \frac{x-1}{(T-t)^{3/2}} - \frac{1}{2} \frac{1}{\sqrt{T-t}} \right)$$

$$\partial_x g = \Phi'(-) \frac{1}{\sqrt{T-t}}$$

$$\partial_{xx} g = \Phi''(-) \frac{1}{T-t} = -\left(\frac{x-1}{\sqrt{T-t}} + \sqrt{T-t} \right) \frac{\Phi'(-)}{T-t}$$

$$\Phi'(y) = \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}$$

$$\Phi''(y) = -y \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}$$

$$= -y\, \Phi'(y)$$

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$$\mathbb{E}\left[\left(\int_0^t g(X_s, s)\, dW_s\right)^2\right]$$

$$= \mathbb{E}\left[\int_0^t g^2(X_s, s)\, ds\right]$$

$$Y_t = \int_0^t W_s^2\, dW_s$$

$$\mathbb{E}_0[Y_t] = 0$$

$$\mathbb{E}_t[Y_s] = \mathbb{E}_t\left[\int_0^t W_u^2\, dW_u + \int_t^s W_u^2\, dW_u\right]$$

$$= \int_0^t W_u^2\, dW_u + \mathbb{E}_t\left[\int_t^s W_u^2\, dW_u\right]$$

$$= Y_t$$

$$\mathbb{E}_0[Y_t^2] = \mathbb{E}\left[\int_0^t (W_s^2)^2\, ds\right] = 3\int_0^t s^2\, ds$$

$$= t^3.$$

$$\mathbb{E}[W_s^4] = s^2\, \mathbb{E}[Z^4] \qquad Z \sim N(0,1)$$

$$= 3s^2$$

$$\int W_s \, dW_s = \int_0^t \frac{1}{2} d(W_s^2) - \frac{1}{2} \int_0^t ds$$

$$= \frac{1}{2}(W_t^2 - W_0^2) - \frac{1}{2} t$$

$$d(W_t^2) = dt + 2W_t \, dW_t$$

$$\int f(W_s) \, dW_s \qquad \text{suppose} \quad g(x) = \int f(x) \, dx$$

then $$d g(W_t) = \frac{1}{2} g''(W_t) \, dt + f(W_t) \, dW_t$$

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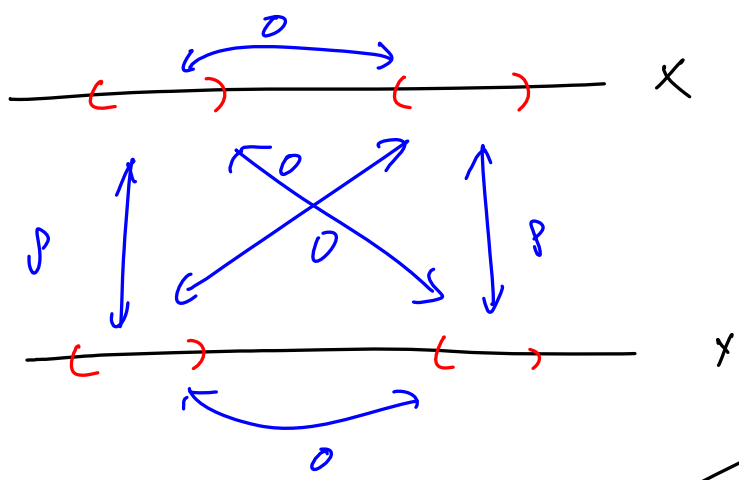
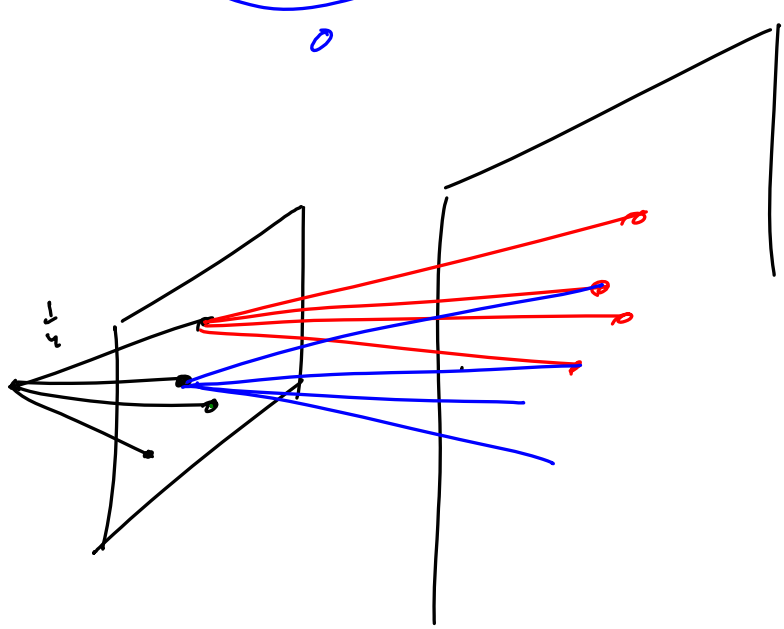
Suppose X_t & Y_t are correlated B. motions. ρ.

$$X_t = Z_t \qquad Z_t, Z_t^{\perp} \text{ independent B. motions.}$$

$$Y_t = \rho Z_t + \sqrt{1-\rho^2}\, Z_t^{\perp}$$

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} t & \rho t \\ \rho t & t \end{pmatrix} \right)$$





$$F(X_t, Y_t)$$

$$dF = \left(\frac{1}{2}\partial_{xx}F + \frac{1}{2}\partial_{yy}f + \rho\,\partial_{xy}f\right)dt$$

$$+ \partial_x f\, dX_t + \partial_y F\, dY_t$$

$$dg_t = \mu^g\, dt + \sigma^g\, dX_t$$
$$dh_t = \mu^h\, dt + \sigma^h\, dY_t$$

$$f(g_t, h_t)$$

$$df = \Big(\mu^g \partial_g f + \mu^h \partial_h F$$

$$+ \frac{1}{2}\big[(\sigma^g)^2 \partial_{gg} f + 2\rho\sigma^g\sigma^h \partial_{gh} F$$

$$+ (\sigma^h)^2 \partial_{hh} f\big]\Big)\, dt$$

$$+ \sigma^g \partial_g f\, dX_t + \sigma^h \partial_h f\, dY_t$$