$$\frac{dU_t}{U_t} = \mu\, dt + \sigma\, dW_t$$

$$\Rightarrow U_s = U_t \exp\left\{ (\mu - \tfrac{1}{2}\sigma^2)(s-t) + \sigma (W_s - W_t) \right\}$$

$$X_t = \ln U_t = f(U_t) \quad , \quad f(u) = \ln u$$

$$dX_t = \underline{\partial_u f\, dX_t} + \underline{\tfrac{1}{2}\sigma^2 U_t^2\, \partial_{uu} f\, dt}$$

$$= \left(\underline{\partial_t + \mu U_t \partial_u} + \underline{\tfrac{1}{2}(\sigma U_t)^2 \partial_{uu}} \right) f\, dt$$

$$+ \underline{\sigma U_t \partial_u f\, dW_t}$$

$$= \left[0 + \mu \cancel{U_t} \frac{1}{\cancel{U_t}} + \tfrac{1}{2}(\sigma \cancel{U_t})^2 \left(-\frac{1}{\cancel{U_t^2}}\right) \right] dt$$

$$+ \sigma \cancel{U_t} \left(\frac{1}{\cancel{U_t}} \right) dW_t$$

$$\Rightarrow dX_t = (\mu - \tfrac{1}{2}\sigma^2)\, dt + \sigma\, dW_t$$

$$X_s - X_t = (\mu - \tfrac{1}{2}\sigma^2)(s-t) + \sigma (W_s - W_t)$$

$$\Rightarrow X_s = X_t + (\mu - \tfrac{1}{2}\sigma^2)(s-t) + \sigma (W_s - W_t)$$

$$\ln U_s = \ln U_t + \quad - \; -$$

$$U_s = U_t \exp\left\{ (\mu - \tfrac{1}{2}\sigma^2)(s-t) + \sigma (W_s - W_t) \right\}$$

$$\frac{dU_t}{U_t} = \mu U_t \, dt + \sigma \, dW_t$$

can still consider $d \ln U_t$

$$dX_t = (\mu U_t - \tfrac{1}{2}\sigma^2)\, dt + \sigma \, dW_t$$

$$X_s - X_t = \int_t^s (\mu U_z - \tfrac{1}{2}\sigma^2)\, dz + \sigma (W_s - W_t)$$

$$U_s = U_t \exp\left\{ \int_t^s (\mu U_z - \tfrac{1}{2}\sigma^2)\, dz + \sigma (W_s - W_t) \right\}$$

$$dU_t = \mu U_t \, dt + \sigma U_t \, dW_t$$

$$U_s - U_t = \mu \int_t^s U_z \, dz + \sigma \int_t^s U_z \, dW_z .$$

$$\frac{dU_t}{U_t} = \mu(U_t, t)\, dt + \sigma(U_t, t)\, dW_t$$

$$U_s = U_t \exp\left\{ \int_t^s \left(\mu(U_z, z) - \tfrac{1}{2}\sigma^2(U_z, z)\right) dz + \int_t^s \sigma(U_z, z)\, dW_z \right\}$$

$$d(\ln U_t) = \left(\mu(U_t, t) - \tfrac{1}{2}\sigma^2(U_t, t)\right) dt + \sigma(U_t, t)\, dW_t$$

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$$dX_t = \kappa(\theta - X_t)\,dt + \sigma\,dW_t$$

θ_t

$$Y_t = X_t - \theta_t$$

$$dY_t = dX_t = \kappa(\underbrace{\theta - X_t}_{-Y_t})\,dt + \sigma\,dW_t \quad - \partial_t\theta\,dt$$

$$= \cancel{-\kappa Y_t\,dt + \sigma\,dW_t} \quad - \partial_t\theta\,dt$$

$- \partial_t \theta$

$$dY_t = -\kappa Y_t\,dt \Rightarrow Y_t = Y_0 e^{-\kappa t}$$

$$Y_t = e^{-\kappa t} g_t$$

$$dY_t = \cancel{-\kappa e^{-\kappa t} g_t\,dt + e^{-\kappa t} dg_t} + d[e^{-\kappa t}, g]_t$$

0

$$\therefore \quad e^{-\kappa t} dg_t = \sigma\,dW_t$$

$$\Rightarrow \quad dg_t = \sigma e^{+\kappa t}\,dW_t$$

$$\Rightarrow \quad g_t - g_0 = \sigma \int_0^t e^{\kappa s}\,dW_s.$$

NB: $\int_0^t e^{\kappa s}\,dW_s \sim N(\cdot, \cdot)$

$$\mathbb{E}\left[\int_0^t e^{\kappa s}\,dW_s\right] = 0$$

$$\mathbb{V}\left[\int_0^t e^{\kappa s}\,dW_s\right] = \mathbb{E}\left[\left(\int_0^t e^{\kappa s}\,dW_s\right)^2\right]$$

$$= \mathbb{E}\left[\int_0^t e^{2\kappa s}\,ds\right]$$

$$= \frac{e^{2\kappa t} - 1}{2\kappa}.$$

$-\kappa t$

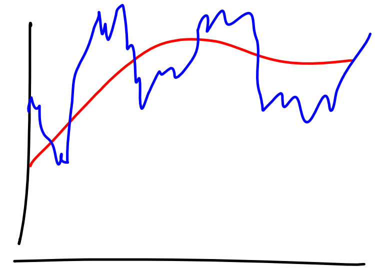
$$Y_t = g_t e$$

$$= g_0 e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)} dW_s$$

$$X_t = \theta + g_0 e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)} dW_s$$

$$g_0 = X_0 - \theta \quad (\text{take } t \downarrow 0)$$

$$\Rightarrow \boxed{X_t = \theta + (X_0 - \theta) e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)} dW_s .}$$



deterministic, diff.

$$dX_t = \kappa(\theta_t - X_t)\,dt + \sigma\, dW_t$$

$$Y_t = X_t - h_t$$

$$dY_t = dX_t - dh_t$$

$$= \kappa(\theta_t - X_t)\,dt + \sigma\, dW_t - \partial_t h\, dt$$

$$= \kappa\left(\theta_t - \frac{\partial_t h}{\kappa} - X_t\right) dt + \sigma\, dW_t$$

$$= \kappa\Big(\underbrace{\Big(\theta_t - h_t - \frac{\partial_t h}{\kappa}\Big)}_{\equiv 0} - Y_t\Big)\, dt + \sigma\, dW_t$$

$$= -\kappa Y_t\, dt + \sigma\, dW_t$$

$$\Rightarrow \quad Y_t = y_0 e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)}\, dW_s$$

$$X_t = h_t + y_0 e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)}\, dW_s$$

$$y_0 = X_0 - h_0 \quad (\text{take } t \downarrow 0)$$

$$X_t = h_t + (X_0 - h_0)\, e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)}\, dW_s$$

choose $h_0 = \theta_0$.

$$\partial_t h_t - \kappa(h_t - \theta_t) = 0$$

$$h_t = e^{+\kappa t} l_t$$

$$\partial_t h_t = + \cancel{\kappa h_t} + e^{-\kappa t}\, \partial_t l_t$$

$$= \cancel{\kappa h_t} - \kappa \theta_t$$

$$\partial_t l_t = -\kappa e^{\kappa t} \theta_t$$

$$l_t - l_0 = -\kappa \int_0^t e^{\kappa s} \theta_s \, ds$$

$$X_t = W_t \underbrace{\int_0^t W_s \, dW_s}_{Y_t} = F(W_t, Y_t)$$

$$dY_t = W_t \, dW_t$$

$$dX_t = \Big(0 + 0 \cdot \partial_w F + \frac{1}{2} \cdot 1^2 \cdot \partial_{ww} F$$

$$+ 0 \cdot \partial_y F + \frac{1}{2} \cdot W_t^2 \cdot \partial_{yy} F$$

$$+ 1 \cdot 1 \cdot W_t \, \partial_{wy} F \Big) dt$$

$$+ 1 \cdot \partial_w F \, dW_t + W_t \cdot \partial_y F \cdot dW_t$$

$$= W_t \, dt + Y_t \, dW_t + W_t^2 \, dW_t$$

$$= W_t \, dt + (Y_t + W_t^2) \, dW_t .$$

$$X_t = W_t Y_t$$

$$dX_t = dW_t \, Y_t + W_t \, dY_t + d[W, Y]_t$$

$$= Y_t \, dW_t + W_t W_t \, dW_t + W_t \, dt$$

$$\left[\int_0^t g_s\,dW_s,\ W_t\right] = \int_0^t g_s\,ds$$

$d[\,,\,]$

$$\sim \left[g_s\,dW_s,\ dW_s\right] \sim g_s\,ds$$

$$\left[\int_0^t g_s\,dW_s,\ \int_0^t h_s\,dW_s\right]$$

$$= \int_0^t g_s h_s\,ds$$

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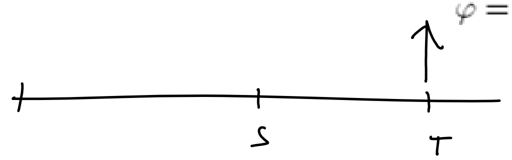
$r=0$

$\varphi = U_T \times \mathbb{I}_{V_S>\gamma}$



$$p_0 = \mathbb{E}^{\mathbb{Q}}\left[U_T \, \mathbb{1}_{V_s>\gamma} \right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}_s\left[U_T \, \mathbb{1}_{V_s>\gamma} \right]\right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{V_s>\gamma} \, \mathbb{E}^{\mathbb{Q}}_s\left[U_T \right]\right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{V_s>\gamma} \, U_s \right]$$

$$U_s \overset{d}{=} U_0 \exp\left\{ -\tfrac{1}{2}\sigma^2 s + \sqrt{s}\,\sigma\left(\rho Z + \sqrt{1-\rho^2}\, Z^\perp\right)\right\}$$

$$V_s \overset{d}{=} V_0 \exp\left\{ -\tfrac{1}{2}\eta^2 s + \sqrt{s}\,\eta\, Z \right\}$$

$$Z, Z^\perp \underset{\mathbb{Q}}{\sim} N(0,1) \text{ and ind.}$$

$$p_0 = \mathbb{E}^{\mathbb{Q}}\Big[U_0\, e^{-\frac{1}{2}\sigma^2 s + \sqrt{s}\,\sigma\, Z^\perp \sqrt{1-\rho^2}} \cdot e^{\sqrt{s}\,\sigma\rho Z}\, \mathbb{1}_{Z>z^*} \Big]$$

$$= U_0\, e^{-\frac{1}{2}\sigma^2 s + \frac{1}{2} s\sigma^2(1-\rho^2)}\ \mathbb{E}^{\mathbb{Q}}\left[e^{\sqrt{s}\,\sigma\rho Z}\, \mathbb{1}_{Z>z^*}\right]$$

$$\hookrightarrow \int_{z^*}^{\infty} e^{\sqrt{s}\,\sigma\rho z}\, e^{-\frac{1}{2}z^2} \frac{dz}{\sqrt{2\pi}}$$

$$= \int_{z^*}^{\infty} e^{-\frac{1}{2}(z-\sqrt{s}\,\sigma\rho)^2 + \frac{1}{2} s\sigma^2\rho^2} \frac{dz}{\sqrt{2\pi}}$$

$$= e^{\frac{1}{2} s \sigma^2 \rho^2} \int_{z^* - \sqrt{s}\,\sigma\rho}^{\infty} e^{-\frac{1}{2} z^2} \frac{dz}{\sqrt{2\pi}}$$

$$= e^{\frac{1}{2} s \sigma^2 \rho^2} \Phi\left(\sqrt{s}\,\sigma\rho - z^*\right)$$

$$\frac{p_0}{u_0} = \mathbb{E}^{Q_u}\left[\mathbb{1}_{V_s > \gamma} \right]$$

$$p_0 = u_0 \, Q_u (V_s > \gamma)$$

$$\frac{dV_t}{V_t} = \overset{\nearrow 0}{r}\, dt + \eta \, d\hat{W}^V_t$$

$$d\tilde{W}^V_t = -\rho \cdot \sigma \, dt + d\hat{W}^V_t$$

$$\Rightarrow \frac{dV_t}{V_t} = \rho \sigma \eta \, dt + \eta \, d\tilde{W}^V_t$$

$$V_s = V_0 \exp\left\{ \left(\rho\sigma\eta - \frac{1}{2}\eta^2\right) s + \eta \left(\tilde{W}^V_s - \tilde{W}^V_0\right) \right\}$$

$$p_0 = u_0 \, Q_u\left(V_0 \, e^{(\rho\sigma\eta - \frac{1}{2}\eta^2)s + \eta\sqrt{s}\, Z^u} > \gamma \right)$$

$$= u_0 \, \Phi(-d_*)$$