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1. Define $\overline{S}(n)$ as the geometric average of the asset's price over the n ordered times $t_0 = 0 < t_1 < t_2 < \ldots < t_n = T$. That is, $\overline{S}(n) := \left(\prod_{j=1}^{n} S(t_j)\right)^{1/n}$. An Arithmetic Asian Call option pays the following at the maturity date T

$$\varphi(S(t_1), \ldots, S(t_n)) = \left(\frac{1}{n}\sum_{m=1}^{n} S(t_m) - K\right)_+ .$$

A Geometric Asian Call option pays the following at the maturity date T

$$\varphi(S(t_1), \ldots, S(t_n)) = \left(\left(\prod_{m=1}^{n} S(t_m)\right)^{1/n} - K\right)_+ .$$

There is no closed form equation for the price of an Arithmetic Asian option; however, the Geometric Asian can be obtained exactly – derive it!

Clearly $\quad S_{t_i} \overset{d}{=} S_{t_{i-1}} \exp\left\{ (r - \tfrac{1}{2}\sigma^2)\Delta t_i + \sigma\sqrt{\Delta t_i}\, Z_i \right\}$

where $\Delta t_i = t_i - t_{i-1}$ and $Z_1, Z_2, .. Z_n$ are iid and

$Z_1 \underset{Q}{\sim} N(0,1)$

So then,

$$S_{t_i} \overset{d}{=} S_0 \exp\left\{ (r - \tfrac{1}{2}\sigma^2) t_i + \sigma \sum_{j=1}^{i} \Delta t_j Z_j \right\}$$

and

$$\bar{S} = (S_{t_1} S_{t_2} .. S_{t_n})^{1/n}$$

$$\overset{d}{=} S_0 \exp\left\{ \frac{(r - \frac{1}{2}\sigma^2)}{n} \sum_{i=1}^{n} t_i + \underbrace{\frac{\sigma}{n} \sum_{i=1}^{n} \sum_{j=1}^{i} \Delta t_j Z_j}_{X} \right\}$$

NB: $\quad X = \frac{1}{n}[\ \Delta t_1 Z_1$

$$+ \Delta t_1 Z_1 + \Delta t_2 Z_2$$
$$+ \Delta t_1 Z_1 + \Delta t_2 Z_2 + \Delta t_3 Z_3$$
$$+$$
$$\vdots$$
$$+ \Delta t_1 Z_1 + \cdots \quad \cdots \quad \cdots + \Delta t_n Z_n \Big]$$

$$= \frac{1}{n}\, n\, \Delta t_1 Z_1 + (n-1)\Delta t_2 Z_2 + \cdots + \Delta t_n Z_n$$

$$= \frac{1}{n} \sum_{i=1}^{n} (n+1-i)\, \Delta t_i\, Z_i$$

and so $X \underset{\mathbb{Q}}{\sim} N(0;\ \bar{\sigma}^2 T)$ where

$$\bar{\sigma}^2 = \frac{\sigma^2}{T} \sum_{i=1}^{n} \left[\left(\frac{n+1-i}{n} \right) \Delta t_i \right]^2$$

$$\therefore\ \bar{S} \overset{d}{=} S \exp\left\{ (\bar{\mu} - \tfrac{1}{2}\bar{\sigma}^2) T + \bar{\sigma}\sqrt{T}\, Z \right\} \qquad Z \underset{\mathbb{Q}}{\sim} N(0,1)$$

where $(\bar{\mu} - \frac{1}{2}\bar{\sigma}^2) T = \dfrac{(r - \frac{1}{2}\sigma^2)}{n} \displaystyle\sum_{i=1}^{n} t_i$

$$\bar{\mu} = \frac{(r - \frac{1}{2}\sigma^2)}{n} \sum_{i=1}^{n} \frac{t_i}{T} + \frac{1}{2}\bar{\sigma}^2$$

$$\therefore\ V_0 = e^{-rT}\, \mathbb{E}^{\mathbb{Q}}\left[\left(S e^{(\bar{\mu} - \frac{1}{2}\bar{\sigma}^2) T + \bar{\sigma}\sqrt{T} Z} - K \right)_+ \right]$$

$$= e^{-(r-\bar{\mu})T}\, \underbrace{e^{-\bar{\mu} T}\, \mathbb{E}^{\mathbb{Q}}\left[\left(S e^{(\bar{\mu} - \frac{1}{2}\bar{\sigma}^2) T + \bar{\sigma}\sqrt{T} Z} - K \right)_+ \right]}$$

exactly B-S with "risk-free rate" of $\bar{u}$
volatility of $\bar{\sigma}$

$$V_0 = e^{-(r-\bar{u})T}\left[S\,\Phi(d_+) - K e^{-\bar{u}T}\,\Phi(d_-)\right]$$

$$d_\pm = \frac{\ln(S/K) + (\bar{u} \pm \frac{1}{2}\bar{\sigma}^2)T}{\bar{\sigma}\sqrt{T}}$$

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2. Suppose you approximate the following sum (which appears in the arithmetic Asian option payoff) by a log-normal distribution:

$$\bar{S} \triangleq \frac{1}{n}\sum_{m=1}^{n} S(t_m) \stackrel{d}{\approx} e^X \to S e^X$$

with X normally distributed. Based on this approximation, derive an analytical expression to approximate arithmetic Asian option prices.

match first two moments ...

$$\mathbb{E}^{\mathbb{Q}}[\bar{S}] = \left(\frac{1}{n}\sum_{m=1}^{n} e^{r t_m}\right) S \triangleq M_1 S$$

$$\mathbb{E}^{\mathbb{Q}}[\bar{S}^2] = \frac{1}{n^2}\sum_{l,m=1}^{n} \mathbb{E}[S_{t_l} S_{t_m}]$$

$$= \frac{1}{n^2}\left(\sum_{l=1}^{n} \mathbb{E}^{\mathbb{Q}}[S_{t_l}^2] + 2\sum_{l<m=1}^{n} \mathbb{E}^{\mathbb{Q}}[S_{t_l} S_{t_m}]\right)$$

$$\mathbb{E}^{\mathbb{Q}}[S_{t_l}^2] = S^2 e^{2(r-\frac{1}{2}\sigma^2)t_l}\, \mathbb{E}^{\mathbb{Q}}[e^{2\sigma\sqrt{t_l}\,Z}]$$

$$= S^2 e^{2(r-\frac{1}{2}\sigma^2)t_l + \frac{1}{2}\cdot 4\sigma^2 t_l}$$

$$= S^2 e^{(2r+\sigma^2)t_l}$$

$$\mathbb{E}^{\mathbb{Q}}[S_{t_l} S_{t_m}] = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}[S_{t_l} S_{t_m} \mid S_{t_l}]]$$

$$= \mathbb{E}^{\mathbb{Q}}[S_{t_l}\, \mathbb{E}^{\mathbb{Q}}[S_{t_m} \mid S_{t_l}]]$$

$$= \mathbb{E}^{\mathbb{Q}}[S_{t_l}\; S_{t_l} e^{r(t_m - t_l)}]$$

$$= e^{r(t_m - t_l)}\, \mathbb{E}^{\mathbb{Q}}[S_{t_l}^2]$$

$$= S^2 e^{r(t_m - t_l) + (2r+\sigma^2) t_l}$$

so that,

$$\mathbb{E}^{\mathbb{Q}}[\bar{S}^2] = \frac{S^2}{n^2}\left(\sum_{l=1}^{n} e^{(2r+\sigma^2)t_l} + 2 \sum_{l<m=1}^{n} e^{r(t_m - t_l) + (2r+\sigma^2)t_l} \right)$$

$$\triangleq M_2 S^2$$

As well, if $X \sim N((\bar{\mu} - \frac{1}{2}\bar{\sigma}^2)T, \bar{\sigma}^2 T)$ then

$$\mathbb{E}^{\mathbb{Q}}[S e^X] = e^{\bar{\mu} T} S \quad \text{and}$$

$$\mathbb{E}^{\mathbb{Q}}[S^2 e^{2X}] = e^{2(\bar{\mu} - \frac{1}{2}\bar{\sigma}^2)T + \frac{1}{2}\cdot 4\bar{\sigma}^2 T} S^2 = e^{(2\bar{\mu} + \bar{\sigma}^2)T}$$

To match moments we need

$$\bar{\mu} = \frac{1}{T}\ln M_1 \quad \& \quad 2\bar{\mu} + \bar{\sigma}^2 = \frac{1}{T}\ln M_2$$

$$\Rightarrow \bar{\sigma}^2 = \frac{1}{T}\left(\ln M_2 - 2\ln M_1\right)$$

$$V_0 \approx e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[(S e^X - K)_+]$$

$$= e^{-(r-\bar{\mu})T}\; e^{-\bar{\mu}T}\,\mathbb{E}^{\mathbb{Q}}[(S e^{(\bar{\mu} - \frac{1}{2}\bar{\sigma}^2)T + \bar{\sigma}\sqrt{T} Z} - K)_+]$$

$Z \overset{\mathbb{Q}}{\sim} N(0,1)$

$$V_0 \approx e^{-(r-\bar{\mu})T}\left[S\,\Phi(d_+) - K e^{-\bar{\mu}T}\,\Phi(d_-) \right]$$

$$d_\pm = \frac{\ln(S/K) + (\bar{\mu} \pm \frac{1}{2}\bar{\sigma}^2)T}{\bar{\sigma}\sqrt{T}}$$

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3. Assuming the Black-Scholes model, determine the value and Delta (for all times $0 < t \leq T$) of contingent claims having the following payoffs at time T:

(a) $S_T \, \mathbb{I}(S_T < K)$

(b) $\min(S_T \; ; \; k \, S_U)$ where $U < T$.

3a) $$V_t = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}\left[S_T \mathbb{1}_{S_T < K} \right]$$

better to use S_t as numeraire asset

$$\frac{V_t}{S_t} = \mathbb{E}^{\mathbb{Q}_S}\left[\frac{S_T \mathbb{1}_{S_T < K}}{S_T} \middle| \mathcal{F}_t \right]$$

$$= \mathbb{Q}_S\left(S_T < K \mid \mathcal{F}_t \right]$$

from class $\quad S_T \overset{d}{=} S_t \exp\{ (r + \tfrac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\, Z \}$

$$Z \underset{\mathbb{Q}_S}{\sim} \mathcal{N}(0,1)$$

$$\Rightarrow V_t = S_t \, \mathbb{Q}_S\left(S_T < K \mid \mathcal{F}_t \right)$$

$$= S_t \, \mathbb{Q}_S\left(Z < \underbrace{\frac{\ln(K/S) - (r + \tfrac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}}_{-d_+} \right)$$

$$= S_t \, \Phi(-d_+)$$

for delta ...

$$\Delta_t = \partial_S V_t = \Phi(-d_+) - S_t \, \Phi'(-d_+) \, \partial_S d_+$$

$$\partial_S d_+ = \frac{1}{S\,\sigma\sqrt{T-t}}$$

$$\Phi'(x) = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$$

5) solved in tutorial.