On Valuing Equity-Linked Insurance and Reinsurance Contracts *

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Through the issuance of equity-linked insurance policies, insurance companies are increasingly facing losses that have heavy exposure to capital market risks. In this paper, we determine the continuous premium rate that an insurer exposed to such risks charges via the principle of equivalent utility. Using exponential utility, we obtain the resulting premium rate in terms of an expectation under the unique minimal martingale measure and perform a perturbation expansion around a risk-neutral investor. Within the same consistent framework, we address the problem of pricing of a double-trigger reinsurance contract, taking into account counter-party risk. The indifference price is found to satisfy a non-linear and nonlocal PDE. This price is further expanded around the risk-neutral price resulting in closed form solutions in the form of risk-neutral expectations. Finally, we recast the pricing PDE as a linear stochastic control problem and provide an implicit-explicit finite-difference scheme for solving the PDE numerically.

1. Introduction

With the $S\&P$ 500 index yielding returns of 7.5% over the last year and 17.8% over the last two years, it is no wonder that individuals seeking insurance are more often opting for equity-linked contracts rather than their fixed payment counterparts. Equity-linked insurance contracts are highly popular options for policyholders because they also provide downside protection in addition to the upside equity like growth potential. From the insurer's perspective, such contracts induce claim sizes that are linked to the fluctuations in the value of the index and, as such, possess significant market risk in addition to the traditional mortality risk. Determining the premium rate for this class of contracts is a daunting task which, due to the non-hedgable nature of the contracts, requires a delicate balancing of the insurer's risk preference, mortality exposure, and

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market exposure. In this work, we adopt the principle of equivalent utility, also known as utility-based or indifference pricing to value such contracts. This pricing principle prescribes a premium rate at which the insurer is indifferent between (i) not taking on the risk and receiving no premium or (ii) taking on the risk while receiving a premium. We review the methodology in more detail at the end of §2. Utility methods are pervasive in incomplete market settings, and were first used by Hodges and Neuberger (1989) to value options subject to transaction costs. This early work was later elaborated by Davis, Panas, and Zariphopoulou (1993) and generalized by Barles and Soner (1998). We continue along this trend but apply it to a new and interesting setting.

Equity-linked life insurance policies have been considered in a few recent works. Young and Zariphopoulou (2002, 2003) were the first to use utility-based methods to price insurance products, albeit with uncorrelated insurance and financial risks. Young (2003), on the other hand, studies equity-linked life insurance policies with a fixed premium and with a death benefit that is linked to an index. She demonstrates, through utility indifference, that the insurance premium satisfies a non-linear Black-Scholes-like PDE where the nonlinearity arises due to the presence of mortality risk. It is well known that insurance risks induce incompleteness into the economy and in such incomplete markets, equivalent utility pricing methods are both useful and powerful. Even when the risky asset itself has non-hedgable jump risks, Jaimungal and Young (2005) demonstrate that the indifference pricing methodology yields tractable and intuitively appealing results. Our work extends these recent studies in two main directions: firstly, by considering equity-linked losses that continually arrive at Poisson times; and secondly, by simultaneously considering the valuation of a reinsurance product within one consistent framework. Such claims process can arise from a large portfolio of equity-linked term life insurance policies, and the reinsurance contract may be purchased on this large portfolio either at policy initiation, or at some future point in time.

We assume that the equity-linked claims (losses) arrive at Poisson times and that the insurer may invest continuously and update her holdings in the equity on which the claims are written. As with all utility-based approaches, this requires a specification of the real world (as opposed to risk-neutral) evolution of equity returns and claim arrivals. We employ the usual geometric Brownian motion assumptions on the index and present our specific modeling assumptions in §2. In §3, we then determine the premium rate for this portfolio of insurance claims through the principle of equivalent utility, focusing exclusively on exponential utility for several well-known reasons: Firstly, the optimal investment strategies dictated by exponential utility is independent of the insurer's wealth. Secondly, the difference between the equity holdings, with and without the insurance risk, reduces to the Black and Scholes (1973) hedge in the complete market case. Thirdly, in the limit in which the investor becomes risk-neutral, the premium reduces to the risk-neutral expected losses over the insurer's investment time horizon.

Through exponential utility, we derive the two Hamilton-Jacobi-Bellman (HJB) equations corresponding to the premium problem and solve them explicitly for any level of risk-aversion and any (reasonably well behaved) equity-linked loss function. We find that the resulting indifference premium q is proportional to the expectation of an exponentially weighted average of the equity-linked loss function under the minimal martingale measure. Interestingly, the premium rate can be reinterpreted as the scaled difference between the expected number of claims in a risk-adjusted measure and the minimal martingale one. For constant losses, we are able to express the premium rate explicitly in terms of the exponential integral function. In the option theoretic framework and in the limit of zero risk-averse investor leads, utility indifference produces prices that are expectations of the payoff under the minimal entropy measure as proved by Becherer (2001). This coincides with the so-called fair price, introduced by Davis (1998), associated with an infinitesimal position in the option. Motivated by these works, we investigate the behavior of a near risk-neutral insurer by performing an asymptotic expansion of the exact result around the zero risk-aversion case. If the claim sizes are positive and bounded by a power of the index level, the series converges uniformly. Interestingly, all terms occurring in the series expansion are expressible as expectations, under the minimal martingale measure, of various powers of the loss function. We also generalized the problem to include the arrival of several loss functions, arising, for example, from claims induced in several groups of policies.

Any insurer who takes on equity-linked insurance risks is exposed to potentially large losses in the event of good market conditions and/or poor underwriting; consequently, in §4, we consider the related problem of pricing double-trigger reinsurance contracts once the insurer has fixed her premium rate. One can view the pricing problem together with the insurance premium problem, or at a later date once the policy rate has been fixed. In either case, the reinsurance contract is assumed to pay a function of the total observed losses, and the equity value, to the insurer at the maturity date. The insurer is assumed to pay an upfront fee (single benefit premium) for this contract – the generalization to periodic payments is not difficult. Utility indifference is once again invoked to determined the value the insurer assigns to the contract. We prove that this price satisfies a Black-Scholes-like PDE with non-linear and nonlocal correction terms due to the presence of the non-hedgable mortality risk. When the reinsurance payoff is independent of the loss level, the indifference price reduces to the Black-Scholes price of the corresponding equity option. This is an appealing result since in that limit the market becomes complete. Purchasing a reinsurance contract leaves the insurer exposed to potentially large counter-party risk; consequently, we model the default time of the reinsurance company as the first arrival time of an inhomogenous Poisson process. This introduces one more non-linear term to the PDE arising in the absence of counter-party risk. Analogous to the expansion of the premium rate around a risk-neutral insurer, we perform an asymptotic expansion of the reinsurance indifference price around a zero risk-aversion parameter. We once again find that the price can be written in terms of expectations under the minimal martingale measure, now with a default-adjusted rate of discount.

In §4.2, we provide a probabilistic interpretation of the indifference price in terms of a dual optimization problem. Within this framework, the indifference price is the minimum of the risk-neutral expected value of the reinsurance contract with a penalty term, where the minimum is computed over the activity rates of the doubly stochastic Poisson processes driving the claim arrivals and counter-party default.

In §4.3, we provide a simple implicit-explicit numerical scheme for the reinsurance contract price. We provide two illustrative prototypical examples: (i) a stop-loss payoff; and (ii) two related double-trigger stop-loss payoffs. The double trigger products are triggered by the asset level rising above or dropping below a critical level. The sum of the two double trigger payoffs results in a pure stop-loss reinsurance contract; however, the sum of their values do not equal to the value of the stop-loss contract. This is indicative of the non-linear pricing rules implied by utility indifference.

2. The modeling and pricing framework

To model the problem for insurers exposed to equity-linked losses, we assume that there is a risky asset whose price process follows a Geometric Brownian motion, and that losses arrive at Poisson times with claim sizes depending on the price of the risky asset at the claim arrival time. More specifically, let $\{S(t)\}_{0\le t\le T}$ denote the price process for a risky asset; let $\{L(t)\}_{0\le t\le T}$ denote the loss process for the insurer; let $\mathcal{F}^S \equiv \{\mathcal{F}^S\}_{0\le t\le T}$ denote the natural filtration generated by $S(t)$; let $\mathcal{F}^L \equiv \{\mathcal{F}^L\}_{0\le t\le T}$ denote the natural filtration generated by $L(t)$; let $\mathcal{F} \equiv \mathcal{F}^S \vee \mathcal{F}^L$ denote the product filtration generated by the pair $\{S(t), L(t)\}$; and let $(\Omega\ ,\ \mathbb{P},\ \mathcal{F})$ represent the corresponding filtered probability space with statistical probability (or real-world) measure $\mathbb{P}$.

The insurer is assumed to invest continuously in the risky asset $S(t)$ and a risk-free money market account with constant yield of $r \ge 0$. Furthermore, the risky asset's price process satisfies the SDE:

$$dS(t) \;=\; S(t)\,\{\mu\, dt + \sigma\, dX(t)\}\ , \tag{1}$$

where $\{X(t)\}_{0\le t\le T}$ is a standard $\mathbb{P}$-Brownian process, and $\mu > r$. Equivalently,

$$S(t) = S(0)e^{\left(\mu-\frac{1}{2}\sigma^2\right)t+\sigma X(t)}\,. \tag{2}$$

The claims arrive at the arrival times $\{t_i\}$ of an inhomogenous Poisson process with deterministic hazard rate $\lambda(t)$, and each claim size is a function of the prevailing index level and possibly time: $g(S(t_i), t_i)$. Notice that the loss size depends on the price of the risky asset prevailing at the time the loss arrives. This is a defining feature of equity-linked insurance products and introduces a new dimension to the optimal stochastic control problem associated with pricing the premium stream. The loss process may be written in terms of an underlying Poisson counting process $\{N(t)\}_{0\le t\le T}$ as follows

$$L(t) = \sum_{n=1}^{N(t)} g(S(t_i), t_i)\,, \tag{3}$$

We implicitly assume that the claims are positive $g(S,t) \ge 0$ and are bounded for every finite pair $(S,t) \in [0,\infty) \times [0,T]$.

Since our assumptions on the dynamics of the risky asset and the loss process have been addressed, we turn attention to the dynamics of the wealth process for the insurer. There are two separate situations of interest: (i) the insurer does not take on the insurance risk, however, the insurer does invest in the risky asset and the riskless money-market account; and (ii) the insurer takes on the insurance risk in exchange for receiving a continuous premium of q and simultaneously invests in the risky asset and the riskless money-market account. Let $\{W(t)\}_{0\le t\le T}$ and $\{W^L(t)\}_{0\le t\le T}$ denote, respectively, the wealth process of the insurer who does not take on the insurance risk (as in case (i)) and the wealth process of the insurer who does take on the insurance risk (as in case (ii)). The process $\pi \equiv \{(\pi(t), \pi_0(t))\}_{0\le t\le T}$ denotes an $\mathcal{F}_t$-adapted self-financing investment strategy, where $\pi(t)$ and $\pi_0(t)$ represent the amount invested in the risky asset and the amount in the money-market account, respectively. The wealth processes then satisfy the following two SDEs:

$$\begin{cases} dW(u) &= [r\,W(u) + (\mu - r)\,\pi(u)]\,du + \sigma\,\pi(u)\,dX(u)\,, \\ W(t) &= w\,, \end{cases} \tag{4}$$

$$\begin{cases} dW^L &= \left[r\,W^L(u_-) + (\mu - r)\,\pi(u_-) + q\right]du + \sigma\,\pi(u_-)\,dX(u) - dL(u)\,, \\ W^L(t) &= w\,, \end{cases} \tag{5}$$

where w represents the wealth of the insurer at the initial time t; and $f(u_-)$ represents the value of the process f prior to any jump at u.

To complete the model setup, we suppose that the insurer has preferences according to an exponential utility of wealth $u(w) = -\frac{1}{\hat\alpha}e^{-\hat\alpha\,w}$ for some $\hat\alpha > 0$. The parameter $\hat\alpha$ is the constant absolute risk-aversion $r_{\hat\alpha}(w) \equiv -u''(w)/u'(w) = \hat\alpha$ as defined by Pratt (1964). Furthermore, the insurer seeks to maximize her expected utility of terminal wealth at the investment time horizon T. This results in two separate stochastic optimal control problems. The value function of the insurer who does not accept the insurance risk is denoted by $V(w,t)$, and the value function of the insurer who does accept the insurance risk is denoted by $U(w,S,t;q)$. Explicitly, the value functions are defined as follows:

$$V(w,t) \;=\; \sup_{\pi\in\mathcal{A}} \mathbb{E}\left[u(W(T))\,|\,W(t)=w\right], \qquad \text{and} \tag{6}$$

$$U(w,S,t;q) \;=\; \sup_{\pi\in\mathcal{A}} \mathbb{E}\left[u(W^L(T))|W^L(t)=w, S(t)=S\right]. \tag{7}$$

Here, $\mathcal{A}$ is the set of admissible, square integrable, and self-financing, $\mathcal{F}_t$-adapted trading strategies for which $\int_t^T \pi^2(s)\,ds < +\infty$. This restriction is necessary for the existence of a strong solution to the wealth process SDEs (4) and (5) (see Fleming and Soner, 1993).

A priori, it is not obvious that V should depend solely on the wealth process and time; similarly, it is not obvious that U should be independent of the loss L. However, through the explicit solutions in the next section, we determine that this is indeed the case – a familiar result when working with exponential utility. Although the value functions are found to depend on the insurer's wealth, the optimal investment strategy is, in fact, independent of the wealth. This too is a consequence of exponential utility.

Next, the indifference premium is defined as the premium q such that the two value functions are equal:

$$V(w,t) = U(w,S,t;q)\,. \tag{8}$$

Intuitively, this implies that the insurer is equally willing either to accept the risk and receive a premium, or to decline the risk and receive no premium.

Once the indifference premium is obtained, the problem of pricing a reinsurance contract is considered in §4. The contract is assumed to pay an arbitrary function $h(L(T), S(T))$ of the total observed losses and the risky asset's price at the time horizon T. The associated value function of an insurer who receives this payment will be denoted $U^R(w, L, S, t; q)$ and is explicitly expressed as

$$U^R(w, L, S, t; q) = \sup_{\pi \in \mathcal{A}} \mathbb{E}[u(W^L(T) + h(L(T), S(T)))|W^L(t) = w, S(t) = S, L(t) = L]\,. \tag{9}$$

Notice that the reinsurance payoff is relevant only at the terminal time, and its role is simply to increase the insurer's wealth by the contract value. Although the contract payoff plays an explicitly role only at maturity, it will induce a feed back into the optimal investment strategy, and consequently, feed back into the value function itself. The value function U^R is well defined for any choice of the premium rate q, and in particular, this rate may be different from the indifference premium rate solving (8). In analogy with the indifference premium of the insurance stream, the indifference price $P(L, S, t)$ of the reinsurance contract is the amount of initial wealth the insurer who receives the reinsurance payment is willing to surrender such that the value function with the reinsurance payment is equal to the value function without the reinsurance payment. That is, the indifference price satisfies the equation:

$$U^R(w - P(L, S, t), L, S, t; q) = U(w, L, S, t; q)\,. \tag{10}$$

A posteriori, the price is found to be independent of wealth for exponential utility. Furthermore, even though the reference value function U^R contains exposure to the insurance risk, we find that the indifference price is independent of the premium that the insurer charges. Since U may be greater, equal, or less than V, the insurer may or may not be in favor of holding the insurance risk in the first place. On the other hand, if the insurer is charing the indifference premium rate implicitly defined by (8), then the indifference price defined in (10) will render the insurer indifferent to taking on the insurance, receiving the premium rate q and simultaneously receiving the reinsurance payoff after giving up initial wealth P. In other words, with q chosen to be the indifference premium, there will be a triple equality: $U^R(w - P, L, S, t; q) = U(w, L, S, t; q) = V(w, t)$. The indifference price P, for an arbitrary premium rate q, may be thought of as a marginal or relative indifference price. Most notably, this price does not coincide with the indifference price of the reinsurance contract in total absence of receiving the insurance premium. This price is similar, but not identical, to the relative indifference price introduced by Musiela and Zariphopoulou (2001, 2004) and employed by Stoikov (2005) in the context of pricing volatility derivatives by adding a small position to an already existing portfolio of options.

To complete the pricing methodology, we consider the effect that counter-party risk plays within this framework. Default of the reinsurer is modeled by the first arrival time τ of a second, stochastically independent, inhomogenous Poisson process $M(t)$ with rate of arrival $\kappa(t)$. The payoff $h(L(T), S(T))$ is now uncertain and must be replaced by $\widehat{h}(L(T), S(T), \tau) \equiv h(L(T), S(T))\, \mathbb{I}(\tau > T)$, and the relevant value function is

$$U^{RC}(w, L, S, t; q) = \sup_{\pi \in \mathcal{A}} \mathbb{E}[u(W^L(T) + \widehat{h}(L(T), S(T), \tau))|W^L(t) = w, S(t) = S, L(t) = L, \tau > t]\,. \tag{11}$$

The indifference price is once again defined through the balancing equation

$$U^{RC}(w - P(L,S,t), L, S, t; q) = U(w, L, S, t; q)\,. \tag{12}$$

We find that counter-party risk serves only to add one additional non-linear, but local, term to the pricing PDE which vanishes as the default intensity reduces to zero.

3. The indifference premium rate problem

Now that the stochastic model for the insurer has been described, and the pricing principle has been specified, we can focus on the details of the pricing problem itself. In the next subsection, the value function without the insurance risk is reviewed. The results of that section are essentially those of Merton (1969). Those results are then used in §3.2 to solve the HJB equation for the insurer exposed to the insurance risk. In §3.3, we determine the indifference premium for a general loss function and provide specific examples. In §3.4, we address the issue of hedging the risk associated with this premium choice.

3.1. The value function without the insurance risk

The value function of the insurer who does not take on the insurance risk is defined in (6), and we now use the dynamic programming principle to determine the optimal investment strategy and the value function itself. Given a particular investment strategy π, we determine that V satisfies the following SDE:

$$dV(W,s) \;=\; \left[V_t + (r\,W + (\mu - r)\,\pi)\,V_w + \tfrac{1}{2}\sigma^2\pi^2\,V_{ww}\right] ds + \pi\sigma\,V_w\,dX\,. \tag{13}$$

The subscripts denote the usual partial derivatives of V, and the time dependence of the various processes are suppressed for brevity. Through the usual dynamic programming principle, V solves the HJB equation:

$$\begin{cases} V_t + r\,w\,V_w + \max_\pi \left[(\mu - r)\,\pi\,V_w + \tfrac{1}{2}\sigma^2\pi^2\,V_{ww}\right] = 0, \\ V(w,T) = u(w). \end{cases} \tag{14}$$

We may assume that the optimal investment is provided by the first order condition, and the Verification Theorem confirms the result. To this end, the optimal investment strategy is

$$\pi^*(t) = -\frac{\mu - r}{\sigma^2}\,\frac{V_w}{V_{ww}}\,. \tag{15}$$

On substituting π^* into (14), V is found to satisfy the PDE

$$V_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{V_w^2}{V_{ww}} + r\,w\,V_w = 0\,. \tag{16}$$

Assuming that

$$V(w,t) = -\frac{1}{\hat{\alpha}} e^{-\alpha(t)\,w + \beta(t)}\,, \tag{17}$$

with $\beta(T) = 0$ and $\alpha(T) = \hat{\alpha}$, the HJB equation reduces to

$$-(\alpha_t + r\alpha)\,w + \beta_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 = 0\,, \tag{18}$$

which must hold for all w and t. Therefore,

$$\alpha(t) = \hat{\alpha}\, e^{r(T-t)} \quad \text{and} \tag{19}$$

$$\beta(t) = -\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 (T-t)\,, \tag{20}$$

resulting in the standard optimal (Merton, 1969) investment of

$$\pi^*(t) = \frac{\mu - r}{\hat{\alpha}\,\sigma^2} e^{-r(T-t)}\,. \tag{21}$$

Since the solution satisfies the requirements of the Verification Theorem, π^* corresponds to the optimal investment strategy for (6), and V, given in (17), is the solution of the original optimal stochastic problem.

3.2. The value function with insurance risk

While assuming the insurance company takes on the insurance risk and receives a premium rate of q, we must solve for the optimal investment and value function U, given in (7). Through straightforward methods, we establish the following HJB equation for the value function U:

$$\begin{cases} 0 = & U_t + (r\,w + q)U_w + \mu\, S\, U_S + \frac{1}{2}\sigma^2\, S^2\, U_{SS} \\ & +\lambda(t)\,(U(w - g(S,t), S, t) - U(w, S, t)) \\ & +\max_\pi \left\{\frac{1}{2}\sigma^2 U_{ww}\,\pi^2 + \pi\left[(\mu - r)U_w + \sigma^2 S(t) U_{ws}\right]\right\}\,, \\ U(w, S, T; q) = & u(w)\,. \end{cases} \tag{22}$$

The nonlocal term appears due to the presence of the Poisson claims, and can be explained by observing that a claim arrives in $(t, t+dt]$ with probability $\lambda(t)\,dt$, causing the wealth to drop by $g(S(t), t)$. At first sight, the presence of this nonlocal term appears to render the problem intractable. However, on closer inspection, we find that the HJB equation can be solved explicitly for arbitrary claims.

Theorem. 3.1 *The solution to the HJB system (22) is*

$$U(w, S, t; q) = V(w, t)\exp\left\{-\,\hat{\alpha}\, q\,\frac{e^{r(T-t)} - 1}{r} + \gamma(S, t)\right\}\,, \tag{23}$$

where

$$\gamma(S(t), t) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \lambda(u)\,\left(e^{\alpha(u)\, g(S(u), u)} - 1\right) du\right]\,, \tag{24}$$

and the process $S(t)$ satisfies the following SDE in terms of the $\mathbb{Q}$-Wiener process $\{\overline{X}(t)\}_{0\le t\le T}$,

$$dS(t) = S(t)\, r\, dt + S(t)\,\sigma\, d\overline{X}(t)\,. \tag{25}$$

Furthermore, the optimal investment strategy is independent of wealth and equals

$$\pi^*(S, t) = \frac{e^{-r(T-t)}}{\hat{\alpha}}\left\{\frac{\mu - r}{\sigma^2} + S\,\gamma_S\right\}\,. \tag{26}$$

Proof. By assuming $U_{ww} < 0$, the first order condition supplies the optimal investment strategy as

$$\pi^*(S,t) = -\frac{(\mu - r)U_w + \sigma^2 S(t) U_{ws}}{\sigma^2 U_{ww}}\,. \tag{27}$$

Substitute this into (22), and make the substitution

$$U(w,S,t;q) = V(w,t;q)\exp\{\gamma(S,t)\}\,, \tag{28}$$

where $V(w,t;q)$ denotes the value function of the insurer who receives a premium rate of q but does not accept the insurance risk. If we denote the wealth process for such an insurer as $W^q(t)$, then $W^q(t) = W(t) + q\,t$. Notice that $V(w,t;0) = V(w,t)$, and that $V(w,t;q)$ satisfies the HJB equation:

$$\begin{cases} 0 = & V_t + (r\,w + q)\,V_w + \max_\pi \left[(\mu - r)\,\pi\,V_w + \frac{1}{2}\sigma^2\pi^2\,V_{ww}\right], \\ V(w,T;q) = & u(w). \end{cases} \tag{29}$$

Through straightforward calculations

$$V(w,t;q) = V(w,t)\exp\left\{-\hat{\alpha}\,q\,\frac{e^{r(T-t)} - 1}{r}\right\} \tag{30}$$

is shown to satisfy (29). Making the substitution (28) into (22), we discover, after some tedious calculations, $V(w,t;q)$ factors out of the problem, and the function $\gamma(S,t)$ satisfies the inhomogeneous linear partial differential equation:

$$\begin{cases} 0 & = \lambda(t)\,(e^{\alpha(t)g(S,t)} - 1) + rS\gamma_s + \frac{1}{2}\sigma^2 S^2\gamma_{SS} + \gamma_t\,, \\ \gamma(S,T) & = 0\,. \end{cases} \tag{31}$$

The Feynman-Kac theorem directly leads to solution (39) from which we observe that $U_{ww} < 0$ so that the maximization term is indeed convex. For reasonably well-behaved loss functions $g(S(t),t)$, the Verification Theorem implies that (39) is the value function for the problem and strategy (27) is optimal. Accordingly, substituting the ansätz (28) into π^* leads to (26). □

3.3. The indifference premium

Now that both value functions V and U are found, an explicit representation of the indifference premium is an easy consequence.

Corollary 3.2 *The insurer's indifference premium rate q is independent of wealth*

$$q(S(t),t) = \frac{r}{e^{r(T-t)} - 1}\,\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \lambda(u)\,\frac{e^{\alpha(u)\,g(S(u),u)} - 1}{\hat{\alpha}}\,du\right]. \tag{32}$$

Proof. The indifference premium rate q is defined as the rate q such that $U(w,S,t;q) = V(w,t)$. Expression (32) then immediately follows from Theorem 3.1. □

Notice that if $\forall u \in (t,T]$, $g(S(u),u) > 0\,\mathbb{Q} - a.s.$, i.e. the claims are almost surely positive, then the indifference premium is positive. Furthermore, the below lemma shows that premium is monotonically increasing in $\hat{\alpha}$, implying that increasing risk-aversion induces an increase in the premium which renders the insurer indifferent to the insurance risk. This intuitively appealing result should be expected of any reasonable pricing rule which incorporates risk-aversion.

Lemma 3.3 *The indifference premium rate q is monotone in the risk-aversion level $\hat{\alpha}$ if the claim size $g(S(t),t)$ is strictly positive.*

Proof. It is easy to check that $p(y) := (e^{y\,x} - 1)\,/y$ is monotonically increasing in y for every $x > 0$. Then, under the positivity assumption of g, for every event $\omega \in \Omega$, we have,

$$\int_t^T \lambda(u)\,\frac{e^{\alpha(u)\,g(S(u),u;\,\omega)} - 1}{\hat{\alpha}} \tag{33}$$

is increasing in $\hat{\alpha}$ and the result follows. □

In analyzing the value function U, we assumed that q was constant; however, on glancing at (32) it can be inferred that q is not a constant, and therefore, our assumptions are false, discrediting the analysis. This initial reaction is premature. The situation is best explained by appealing to the familiar case of a forward contract. On signing of a forward contract, the delivery price is set such that the contract has zero value. This delivery price is a function of the prevailing spot price of the asset and bond prices at the time of signing. Although the contract value on signing is zero, the forward price, at any future date, will not equal to the delivery price, and the contract's value is no longer zero. In the present context, the insurer is looking forward to a future time horizon, and is deciding on a rate to charge so that she is indifferent to taking the risk. Our analysis shows that the rate (32), which depends on the prevailing price of the risky asset, should be charged. This rate is fixed until the end of the time horizon, and does indeed render the insurer indifferent to the insurance risk *at the current time*. However, as time evolves, the prevailing indifference premium at that future point in time may be higher or lower than the rate the insurer initially set. Consequently, if the insurer took on the insurance risk at time t in exchange for $q(S(t),t)$ until the horizon end, then at some future time she may develop a preference either towards releasing the insurance risk or for holding onto it. With the forward contract analogy, it is no surprise then that the premium rate depends on the risky asset's spot price.

The indifference premium (32) has some additional noteworthy properties. The risk-neutral measure $\mathbb{Q}$ appearing in the premium calculation is independent of the risk-aversion level of the insurer. Within this risk-neutral measure, the distribution of claim sizes has not been distorted from its real world distribution. Indeed, the Radon-Nikodym derivative process which performs the measure change is

$$\eta(t) \equiv \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_t = \exp\left\{-\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t + \frac{\mu - r}{\sigma}\,X(t)\right\}. \tag{34}$$

This is the same measure change that Merton (1976) uses in his jump-diffusion model and corresponds to risk-adjusting only the diffusion component. Since the equity risk and the losses are interrelated only through the loss size, and not through any instantaneous correlation, this risk-neutral measure corresponds to the minimal martingale one. This is similar to the case of indifference pricing for financial options, where only the tradable asset's risk process are distorted such that their drift is the risk-free one under the pricing measure, while orthogonal stochastic degrees of freedom are left undistorted. Even though the risk-aversion level does not feed into the probability measure used for computing expectations, it does manifest itself in the distortion of the claim sizes through the exponential term. This exponential distortion is, not surprisingly, inherited from the choice of utility function. A particularly interesting interpretation of the premium is supplied by splitting the expectation term appearing in (32) into two and writing each term as the expected number of claims under two different measures. Specifically,

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \lambda(u)\,\left(e^{\alpha(u)\,g(S(u),u)}-1\right)\,du\right] &= \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \lambda(u)\,e^{\alpha(u)\,g(S(u),u)}\,du\right]-\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \lambda(u)\,du\right] \\ &= \mathbb{E}_t^{\widetilde{\mathbb{Q}}}\left[N(T)\right]-\mathbb{E}_t^{\mathbb{Q}}\left[N(T)\right]\,. \end{aligned} \tag{35}$$

Under the measure $\widetilde{\mathbb{Q}}$, the process $N(t)$ is a doubly stochastic Poisson process with activity rate $\widetilde{\lambda}(t)\equiv\lambda(t)\,e^{\alpha(t)\,g(S(t),t)}$. When interest rates are zero and the losses themselves are constant, it is easy to check that the measure $\widetilde{\mathbb{Q}}$ is the minimizer of the penalized entropy:

$$\min_{\widetilde{\mathbb{Q}}<<\mathbb{P}} \mathbb{E}_t^{\widetilde{\mathbb{Q}}}\left[\ln\left(\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}\right)_T - \hat{\alpha}\,g\,N(T)\right]\,. \tag{36}$$

As a final point of interest, although the premium is a non-linear functional of the claim sizes it is linear in the arrival rate of the claims $\lambda(t)$. This observation suggests that the generalization to multiple claims distributions is straightforward. In the theorem below, we provide the results for multiple claims distributions. The proof is omitted for brevity as it follows along the same lines as those in the previous two sections.

Theorem. 3.4 *Suppose that the insurer is exposed to losses from m different sources of risk. Explicitly, the loss process is modeled as follows:*

$$L(t)=\sum_{j=1}^{m}\sum_{n=1}^{N_j(t)} g_j(S(t_j^n),t_j^n)\,, \tag{37}$$

where $\{N_j(t):j=1,\dots,m\}$ are independent inhomogenous Poisson processes with arrival rates $\{\lambda_j(t):j=1,\dots,m\}$, $g_j(S,t)$ denotes the loss functions for the j-th source of risk, and t_j^n denotes the n-th arrival time for the j-th process. Then, the value function of the insurer who takes on the insurance risk and receives a premium of $q(w,S,t)$ is

$$U(w,S,t;q) \;=\; V(w,t)\exp\left\{-\hat{\alpha}\,q\,\frac{e^{r(T-t)}-1}{r}+\gamma(S,t)\right\}\,, \tag{38}$$

where

$$\gamma(S(t),t)=\sum_{j=1}^{m}\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \lambda_j(u)\,\left(e^{\alpha(u)\,g_j(S(u),u)}-1\right)\,du\right]\,, \tag{39}$$

and the process $S(t)$ satisfies the following stochastic differential equation in terms of the $\mathbb{Q}$-Wiener process $\{\overline{X}(t)\}_{0\le t\le T}$:

$$dS(t)=S(t)\,r\,dt+S(t)\,\sigma\,d\overline{X}(t)\,. \tag{40}$$

Furthermore, the insurer's indifference premium is independent of wealth and is explicitly

$$q(w,S,t) = \frac{r}{e^{r(T-t)}-1} \sum_{j=1}^{m} \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^T \lambda_j(u) \, \frac{e^{\alpha(u)\, g_j(S(u),u)}-1}{\hat{\alpha}} \, du \right] . \tag{41}$$

This linearity is an interesting consequence of the continual arrival of claims at Poisson times. If instead, we assumed that there was a maximum finite number of claims, then the problem would not be linear. We are currently investigating how a large, but finite, number of claims alters the results in this article.

3.3.1. Constant losses

An interesting testing ground for our results is the case when the losses themselves are constant, i.e. $g(S,t) = l$. This case is a particular example of the model in Young and Zariphopoulou (2003) where the claims distribution is a Dirac delta distribution; however, they did not report the result for this simple constant claims case. We determine the indifference premium rate as

$$q = \frac{\lambda}{\hat{\alpha}\left(e^{r(T-t)}-1\right)} \left(Ei(\hat{\alpha}\, l\, e^{r(T-t)}) - Ei(\hat{\alpha}\, l) - (T-t)r \right) , \tag{42}$$

where $Ei(x)$ denotes the so called "exponential integral", defined as the following Cauchy principle value integral:

$$Ei(x) \equiv \fint_{-\infty}^{x} \frac{e^t}{t}\, dt\,. \tag{43}$$

The exponential integral has the following asymptotic expansion:$Ei(x) = \gamma + \ln(x) + \sum_{n=1}^{\infty} \frac{x^n}{n!\,n}$; where γ is Euler's constant.

As expected, the indifference premium is non-linear in the claim size l. There is good evidence that insurers who have well diversified portfolios and large reserves exhibit near risk-neutral behavior. It is therefore interesting to investigate the impact this has on the valuation of the insurance stream. If the insurer is near risk-neutral, then an expansion in $\hat{\alpha}\, l$ can be carried out, and we find the indifference premium rate to linear order is

$$q = \lambda\, l \left(1 + \frac{1}{4}\left(e^{r(T-t)} + 1 \right) \hat{\alpha}\, l + o(\hat{\alpha}\, l) \right) . \tag{44}$$

As such, a risk-neutral insurer exposed to fixed losses, will charge a rate equal to the expected loss per unit time $\lambda\, l$ – an intuitively sound result. As expected, the sign of the first order correction is positive.

3.3.2. Near risk-neutral insurer

Extending the near risk-neutral insurer analysis to general claims function is not difficult. For losses that grow at most power like, i.e. there exists $c > 0$, $b(t) > 0$ and $S^*(t) > 0$ such that for each t and $S > S^*(t)$, $g(S,t) \le b(t)\, S^c$, the rate has the following perturbative expansion in terms of the risk-aversion parameter $\hat{\alpha}$:

$$q \;=\; \frac{\lambda\, r}{e^{r(T-t)}-1} \sum_{n=1}^{\infty} \frac{\hat{\alpha}^{n-1}}{n!} \int_t^T e^{n\, r(T-u)}\, \mathbb{E}_t^{\mathbb{Q}}\left[g^n(S(u),u) \right] du\,. \tag{45}$$

This series converges by appealing to the Lebesgue dominated convergence theorem and noting that $\mathbb{E}_t^{\mathbb{Q}}[S^n(u)] = S(t)e^{n(r+\frac{1}{2}\sigma^2(n-1))(u-t)}$. This growth condition can be weakened considerably; however, at this point we are concerned with aiding intuition and as a result, omit such details from the analysis. Expansions similar to the one above have been explored in Davis (1998) in the options context. He demonstrates that the zeroth order term is equivalent to price of an infinitesimal position in the option – the so called marginal price. Based on (45), a risk-neutral insurer would then charge a premium rate of

$$q = \frac{r}{e^{r(T-t)} - 1} \int_t^T \lambda(u)\, e^{r(T-u)} \mathbb{E}_t^{\mathbb{Q}} \left[\, g(S(u), u)\right] du\,. \tag{46}$$

Observe that the factor in front of the expectation can be represented as $(\overline{a})^{-1}$ with $\overline{a} = \int_t^T e^{r(T-u)}\, du$. The expression $\overline{a}$ is precisely the accumulated value of \$1 per annum received continuously over the time span $(t, T]$. Furthermore, the expectation can be interpreted as the risk-neutral expected claims accumulated to the maturity date T. With these points in mind, the risk-neutral indifference premium (46) rate balances, in expectation, between continually receiving the premium and paying out the claims.

3.3.3. Floor, capped, and market participation claims

In this section we provide an explicit example of the premium when the losses are functions of the logarithm of the stock index. While still maintaining the essential properties of linear claim sizes, we use the logarithm of the stock price because it allows for partially closed form solutions. To this end, define $A(u)$ as the expectation appearing under the integral in the indifference premium (32), i.e.

$$A(u) \equiv \mathbb{E}_t^{\mathbb{Q}} \left[e^{\alpha(u)\, g(S(u),u)} \right]\,. \tag{47}$$

Then, the indifference premium q can be written in terms of $A(u)$ explicitly as

$$q = \frac{r}{\hat{\alpha}\left(e^{r(T-t)} - 1\right)} \int_t^T \lambda(u) \left\{ A(u) - 1 \right\} du\,. \tag{48}$$

Consider insurance claims which have a cap and a floor protection in addition to a participation in the risky asset's return independent of time – typical features found in equity-linked insurance payoffs. In this case, the claim sizes are

$$g(S(t), t) = \begin{cases} \theta\,, & S(t) < c_1\,, \\ \theta + \beta\left(\log\left(S(t)\right) - \log(c_1)\right)\,, & c_1 \le S(t) < c_2\,, \\ \theta + \beta\left(\log(c_2) - \log(c_1)\right)\,, & S(t) \ge c_2\,. \end{cases} \tag{49}$$

To maintain positivity of the claim sizes in all outcomes, we restrict $\theta > 0$, $\beta > 0$ and $c_1 < c_2$. After some tedious calculations, the integrand $A(u)$ reduces to

$$\begin{aligned} A(u) = \;& e^{\alpha(u)\theta} \left\{ \Phi(d_1(c_1)) + \left(\tfrac{c_2}{c_1}\right)^{\beta\,\alpha(u)} \Phi(-d_1(c_2)) \right. \\ & \left. + \left(\tfrac{S(t)}{c_1}\right)^{\beta\,\alpha(u)} e^{\beta\,\alpha(u)\left(r - \frac{1}{2}\sigma^2(1-\beta\,\alpha(u))\right)(u-t)} \left(\Phi(d_1(c_3)) - \Phi(d_2(c_1))\right) \right\}. \end{aligned} \tag{50}$$

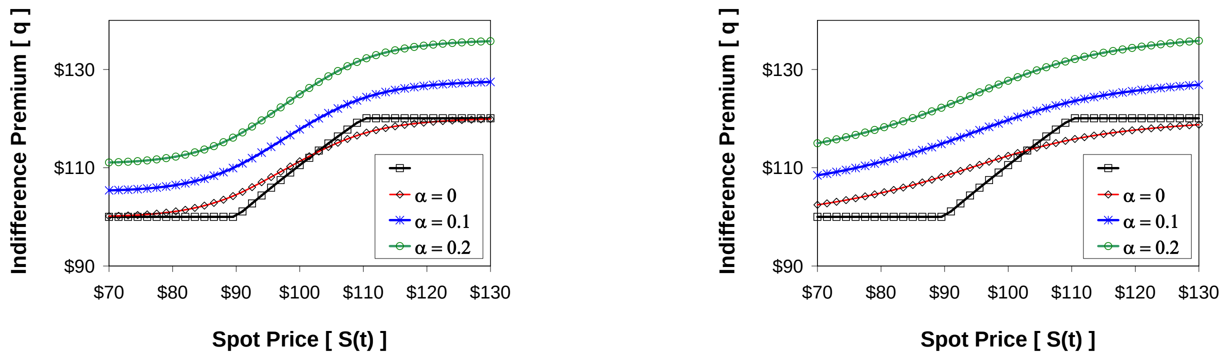

Figure 1. The dependence of the indifference premia on the underlying equity spot price for losses given in equation (49). The model parameters are $\theta = 1$, $\beta = 1$, $c_1 = 90$, $c_2 = 110$, $r = 4\%$, $\sigma = 15\%$, and $\lambda = 100$. The terms in the left/right panels are one and five years respectively.

In Figure 1, the dependence of the premium on the underlying spot price is illustrated for three choices of the risk-aversion parameter $\hat{\alpha}$ and for terms of one and five years respectively. The boxed line shows the pure loss function (49) scaled by the activity rate for comparison purposes. As the risk-aversion parameter increases, the premia increases illustrating the monotonicity property. Notice that when the term increases the premium decreases for large spot prices, while it increases for small spot prices. This is analogous to the pricing for a standard bull-spread option in the Black-Scholes model.

3.4. Hedging the insurance risk

Now that we have determined the indifference premium that the insurer charges, it is interesting to explore the hedging strategy that she would follow. In this incomplete market setting, it is impossible to replicate the insurance claims; nonetheless, the insurer holds different units of the risky asset when she is exposed to the insurance risk and when she is not exposed to the insurance risk. As a result, we can define an analog of the Black-Scholes Delta hedging parameter. To this end, the Delta is defined as the excess units of the risky asset that the insurer holds when taking on the risk and receiving the premiums, and when there is an absence of insurance risk.

Corollary 3.5 *The Delta of the insurer's position is*

$$\Delta(S,t) \equiv \frac{1}{S}\left(\pi_U^* - \pi_V^*\right) \quad = \quad \frac{e^{-r(T-t)}}{\hat{\alpha}}\,\gamma_S(S,t)\,. \tag{51}$$

Proof. The optimal investment in the risky asset without the insurance risk appears in (15), and with the insurance risk appears in (26). □

The result is quite similar to the Black-Scholes Delta for an option. Although it is possible to rewrite the Delta in terms of the indifference premium rate q, it is most naturally represented in terms of the auxiliary function γ. Moreover, as $t \to T^-$, the Delta vanishes; this is quite different from the behavior of the Black-Scholes Delta of an option with payoff $g(S(T),T)$. In the case of

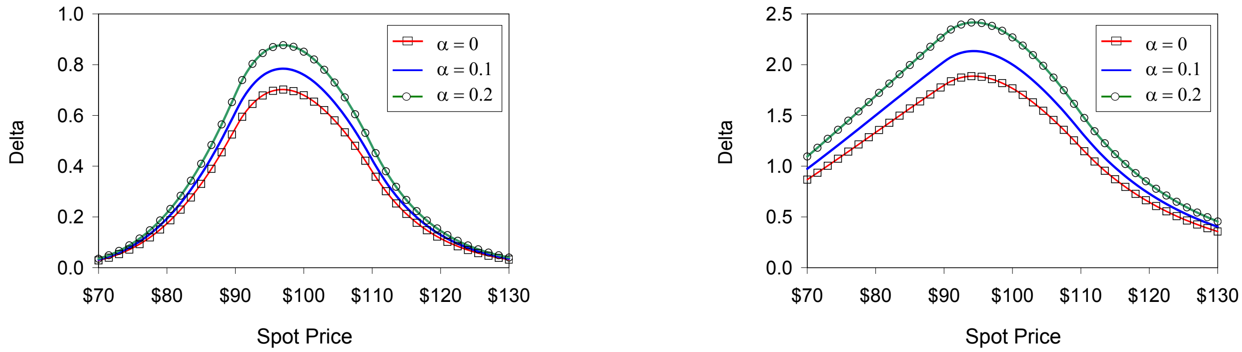

Figure 2. The dependence of the Delta on the underlying equity spot price for losses given in equation (49). The model parameters are $\theta = 1$, $\beta = 1$, $c_1 = 90$, $c_2 = 110$, $r = 4\%$, $\sigma = 15\%$, and $\lambda = 100$. The terms in the left/right panels are one and five years respectively.

a European option, the Delta becomes equal to the derivative of the payoff function with respect to the spot price, and is zero only where the option's payoff becomes flat. In the present context, the Delta vanishes as maturity approaches because the probability of a loss arriving in the next small time interval close to maturity is $\lambda\Delta T$. Therefore, probabilistically, there is no need to hold additional shares of the risky-asset near maturity.

In Figure 2, we show how the Delta behaves as a function of the spot-level, risk-aversion parameter, and time to maturity for the example in §3.3.3. The general shape of these curves is expected. The payoff is asymptotically flat outside of the participation region (see Figure 1); implying a decaying Delta in the tails. The Delta is wider when maturity is further away and, contrary to an option's delta, it is increases with maturity due to the larger number of potential losses.

4. The indifference price for reinsurance

Now that we have determined the indifference premium that the insurers charges, we can address the dual problem of pricing a reinsurance contract which makes payments at the end of the time horizon. In section 2, we describe the value function associated with the insurer who takes on the insurance risk, receives the premium rate q, and receives a reinsurance payment of $h(L(T), S(T))$. The value function of such an insurer was denoted U^R as defined in equation (9). The associated HJB equation for this value function is essentially the same as the one for U (see equation (22)); however, the boundary condition is now altered to account for the presence of the reinsurance, and we must also keep track of the loss process explicitly. Through the usual dynamic programming principle, we determine that U^R satisfies the following HJB equation:

$$\begin{cases} 0 = U^R_t + (rW + q)U^R_w + \mu\, S\, U^R_S + \frac{1}{2}\sigma^2\, S^2\, U^R_{SS} \\ \qquad +\lambda(t)\left(U^R(w - g(S,t), L + g(S,t), S, t) - U^R(w, L, S, t)\right) \\ \qquad + \max_\pi \left\{\frac{1}{2}\sigma^2 U^R_{ww}\,\pi^2 + \pi\left[(\mu - r)U^R_w + \sigma^2 S(t) U^R_{ws}\right]\right\}, \\ U^R(w, L, S, t; q) = u(w + h(L, S)). \end{cases} \tag{52}$$

The nonlocal term now contains two types of shifting: the first, due to the decrease in the wealth of the insurer; and the second, due to the increase in the loss process. However, both shifts come from the same risk source. We can immediately address the issue of counter-party risk by modifying the payoff to include an indicator of the event that the reinsurer survives to the maturity date. As described at the end of section §2, the reinsurer's default time is modeled by a separate inhomogeneous Poisson process $M(t)$ with activity rate $\kappa(t)$. It is easy to see that the value function in the presence of counter-party risk, which we denoted by U^{RC}, satisfies the HJB:

$$\begin{cases} 0 = U_t^{RC} + (rW+q)U_w^{RC} + \mu\, S\, U_S^{RC} + \frac{1}{2}\sigma^2\, S^2\, U_{SS}^{RC} \\ \qquad +\lambda(t)\left(U^{RC}(w-g(S,t), L+g(S,t), S, t) - U^{RC}(w,L,S,t)\right) \\ \qquad +\kappa(t)\left(U(w,L,S,t) - U^{RC}(w,L,S,t)\right) \\ \qquad +\max_\pi\left\{\frac{1}{2}\sigma^2 U_{ww}^{RC}\,\pi^2 + \pi\left[(\mu-r)U_w^{RC} + \sigma^2 S(t) U_{ws}^{RC}\right]\right\}, \\ U^R(w,L,S,t;q) = u(w+h(L,S))\,. \end{cases} \tag{53}$$

The only difference between (52) and (53) is the presence of the nonlocal term proportional to the default intensity $\kappa(t)$. Its appearance can be understood as follows: if a default occurs over the next infinitesimal time, the value function U^{RC} reverts to U since the insurer will no longer receive the reinsurance payment at maturity; however, she is still exposed to the insurance risks and still receives the premium rate q.

Since we can recover the default-free case by setting $\kappa(t) = 0$, the remaining analysis focus only on solving equation (53). Once again, exponential utility allows us to obtain a solution of the HJB equation in a semi-explicit form.

Theorem. 4.1 *The solution to the HJB system (53) can be written as*

$$U^{RC}(w,L,S,t) = U(w,S,t)\phi(L,S,t)\,, \tag{54}$$

where ϕ satisfies the non-linear PDE

$$\begin{cases} 0 = \phi_t + r\,S\phi_S + \frac{1}{2}\sigma^2 S^2\left(\phi_{SS} - \frac{\phi_S^2}{\phi}\right) + \kappa(t)\left(1-\phi(L,S,t)\right) \\ \qquad +\lambda(t)\, e^{\alpha(t)g(S,t)}\left(\phi(L+g(S,t),S,t) - \phi(L,S,t)\right), \\ \phi(L,S,T) = e^{-\hat\alpha\, h(L,S)}\,. \end{cases} \tag{55}$$

Furthermore, the optimal investment in the risky-asset is

$$\pi^*(S,t) = \frac{e^{-r(T-t)}}{\hat\alpha}\left\{\frac{\mu-r}{\sigma^2} + S\left[\gamma_S + \frac{\phi_S}{\phi}\right]\right\}. \tag{56}$$

Proof. Assuming that $U_{ww}^{RC} < 0$, the first order conditions allow the optimal investment strategy to be written,

$$\pi^*(t) = -\frac{(\mu-r)U_w^{RC} + \sigma^2 S(t) U_{wS}^{RC}}{\sigma^2 U_{ww}^{RC}}\,. \tag{57}$$

On substituting the ansätz (54) and the optimal investment (57) into the HJB equation (52), we establish

$$\begin{cases} 0 = & \phi\left\{U_t + (rw+q)U_w + \mu S U_S + \frac{1}{2}\sigma^2 S^2 U_{SS} - \frac{1}{2}\frac{\left((\mu-r)U_w + \sigma^2 S U_{wS}\right)^2}{\sigma^2 U_{ww}}\right\} \\ & +U\left\{\phi_t + \left(\mu - (\mu-r)\frac{U_w^2}{U\,U_{ww}}\right) S\,\phi_S + \frac{1}{2}\sigma^2 S^2\left(\phi_{SS} - \frac{U_w^2}{U\,U_{ww}}\frac{\phi_S^2}{\phi} - 2\left(\frac{U_{wS}U_w}{U\,U_{ww}} - \frac{U_S}{U}\right)\phi_S\right)\right\} \\ & +\kappa(t)\,U(w,S,t)\,\left[1-\phi(L,S,t)\right] \\ & +\lambda(t)\,\left[U(w-g(S,t),S,t)\,\phi(L+g(S,t),S,t) - U(w,S,t)\,\phi(L,S,t)\right]\,, \end{cases} \tag{58}$$

subject to the boundary condition $U(w,S,T)\phi(L,S,T) = u(w+h(L,S))$. From (22), the terms inside $\{\cdot\}$ in the first line of the above expression equals $-\lambda(t)\,[U(w-g(S,t),S,t) - U(w,S,t)]$; collecting this with the last line and making use of the identities

$$U(w-g(S,t),S,t) = U(w,S,t)\,e^{\alpha(t)g(S,t)}\,, \qquad \frac{U_w^2}{U_{ww}U} = 1\,, \quad \text{and} \quad \frac{U_{wS}U_w}{U_{ww}U} = \frac{U_S}{U} = \gamma_S\,, \tag{59}$$

we find, then, equation (58) distills to (55). It can then be proven that $U_{ww}^R < 0$. Using the ansätz (54), the optimal investment π^* can be rewritten as (56). For smooth g, the Verification Theorem allows us to confirm that the constructed solution is the value function for the original problem and that the described strategy is clearly optimal. □

Corollary 4.2 *The insurer's indifference price $P(L(t),S(t),t)$ for the reinsurance contract satisfies the nonlinear nonlocal PDE:*

$$\begin{cases} r\,P = P_t + r\,S\,P_S + \frac{1}{2}\sigma^2\,S^2\,P_{SS} - \frac{\kappa(t)}{\alpha(t)}\left(1 - e^{-\alpha(t)\,P(L,S,t)}\right) \\ \qquad + \frac{\lambda(t)}{\alpha(t)} e^{\alpha(t)g(S,t)}\left(1 - e^{-\alpha(t)[P(L+g(S,t),S,t) - P(L,S,t)]}\right)\,, \\ P(L,S,T) = h(L,S)\,. \end{cases} \tag{60}$$

Proof. The indifference price P satisfies $U^{RC}(w-P,L,S,t) = U(w,S,t)$. The factorization (54) together with the identity $U(w-g(S,t),S,t) = U(w,S,t)\,e^{\alpha(t)g(S,t)}$ implies that $P(L,S,t) = -\frac{1}{\alpha(t)}\ln\phi(L,S,t)$. On substituting ϕ in terms of P in (55), we obtain (60). □

Notice that if the payoff function $h(L,S)$ is independent of the loss level, i.e. $h(L,S) = h(S)$, then (60) reduces to

$$\begin{cases} r\,P &= P_t + r\,S\,P_S + \frac{1}{2}\sigma^2\,S^2\,P_{SS} - \frac{\kappa(t)}{\alpha(t)}\left(1 - e^{-\alpha(t)\,P}\right)\,, \\ P(L,S,T) &= h(S)\,. \end{cases} \tag{61}$$

When there is no counter-party risk, the above pricing equation is precisely that of a European option with payoff $h(S)$ in the Black and Scholes (1973) model. This result is expected since the reinsurance contract is then exposed only to the hedgable risk – the risky asset – and not to the non-hedgable claims risk or counter-party risk. Therefore, our result should reduce to the no arbitrage Black-Scholes price for an insurer of any degree of risk-aversion.

When counter-party risk is the only non-hedgable risk, i.e. $\kappa(t) \neq 0$, but h is a function only of S, then the pricing equation is identical to the indifference pricing equation for an equity-linked pure endowment paying $h(S)$ conditional on survival to maturity as studied in Moore and Young (2003). Alternatively, this price may be viewed as the value of a defaultable option with payoff $h(S)$.

4.1. Near risk-neutral insurer

Let the price of a risk-neutral insurer, taken as the limit of a risk-averse insurer, be denoted $P^0(L,S,t) = \lim_{\hat{\alpha}\to 0^+} P(L,S,t)$. Then, the pricing PDE for P^0 following from (60) is

$$\begin{cases} (r+\kappa(t))\, P^0 = P^0_t + r\, S\, P^0_S + \frac{1}{2}\sigma^2\, S^2\, P^0_{SS} + \lambda(t)\Delta P^0 \\ P^0(L,S,T) = h(L,S)\,, \end{cases} \tag{62}$$

where ΔP^0 denotes the increase in the price due to a loss arrival:

$$\Delta P^0(L,S,t) \equiv P^0(L+g(S,t),S,t) - P^0(L,S,t)\;. \tag{63}$$

Consequently, through the Feynman-Kac Formula, a risk-neutral insurer would be willing to pay

$$P^0(L,S,t) = \mathbb{E}^{\mathbb{Q}}_t\left[e^{-\int_t^T (r+\kappa(s))ds} h(L(T),S(T))\right] = \mathbb{E}^{\mathbb{Q}}_t\left[e^{-\int_t^T r\,ds}\, h(L(T),S(T))\,\mathbb{I}(\tau > T)\right] \tag{64}$$

for the reinsurance contract, where the $\mathbb{Q}$-dynamics of $S(t)$ appears in (40), while the loss arrival rate and counter-party default rate are both unaltered from their real world values. Although this market is incomplete, and therefore there exists many risk-neutral measures equivalent to the real world measure (Harrison and Pliska, 1981), the indifference pricing methodology selects a *unique* measure. This measure is the minimal martingale measure under which stochastic degrees of freedom that are orthogonal to the driving diffusion of the tradable asset's price process remain undistorted.

It is interesting to investigate the first order correction in the risk-aversion parameter $\hat{\alpha}$ to gain some understanding of the perturbations around the risk-neutral price. This is similar to the work of Stoikov (2005) where he investigates the linear corrections of the price of volatility derivatives when the investor has already taken positions in a portfolio of derivatives. In our context we are pricing the reinsurance option in the presence of the insurance risk. The main difference here is that our reinsurance payoff is not considered small relative to the "background portfolio" as in Stoikov (2005).

If we assume that the payoff function is bounded from above, and hence the price is also bounded, then the price can be expanded in a power series in $\hat{\alpha}$. Specifically, write

$$P(L,S,t) = P^0(L,S,t) + \hat{\alpha}P^1(L,S,t) + o(\hat{\alpha})\,, \tag{65}$$

subject to $P^0(L,S,T) = h(L,S)$ and $P^1(L,S,T) = 0$. The linear correction vanishes at maturity since we have fully accounted for the payoff in the zeroth order term. When inserting this ansätz into (60) and using (62), we determine $P^1(L,S,t)$ satisfies the following PDE:

$$\begin{cases} (r+\kappa(t))\, P^1 = P^1_t + r\, S\, P^1_S + \frac{1}{2}\sigma^2\, S^2\, P^1_{SS} + \lambda(t)\Delta P^1 \\ \qquad\qquad + e^{r(T-t)}\left(\frac{1}{2}\kappa(t)\left(P^0\right)^2 + \lambda(t)\left\{g^2(S,t) - \left[\Delta P^0(L,S,t) - g(S,t)\right]^2\right\}\right) + o(\hat{\alpha})\,, \\ P^1(L,S,T) = 0\,. \end{cases} \tag{66}$$

Through Feynman-Kac, the first order correction can be represented as a risk-neutral expectation as well, and we find the following result:

$$P^1(L,S,t) \;=\; \mathbb{E}^{\mathbb{Q}}_t\left[\int_t^T e^{-\int_t^u \kappa(s)\,ds}\left[\frac{1}{2}\kappa(u)\left(P^0(L(u),S(u),u)\right)^2\right)\right.$$

$$+\lambda(u)\left\{g^2(S(u),u)-\left[\Delta P^0(L(u),S(u),u)-g(S(u),u)\right]^2\right\}\Big]\,du\Big]\;.(67)$$

Interestingly, the payoff function $h(L,S)$ does not explicitly appear in P^1; rather, it feeds from the risk-neutral price function P^0 which does explicitly depend on the payoff. The sign of this correction term is difficult to discern on first observation due to the term proportional to $\lambda(u)$. However, we may deduce that if (i) h is increasing in L, (ii) g is non-negative, and (iii) h is Lipschitz-continuous with Lipschitz constant 2, then the correction term is non-negative.

4.2. Probabilistic interpretation of the indifference price

Although explicit solutions to the general pricing PDE (60) were not constructed, we follow Musiela and Zariphopoulou (2003) and show that the price function solves a particular stochastic optimal control problem. By using the convex dual of the non-linear term, the PDE is linearized and results in a pricing result similar to the American option problem. However, in the current context, the optimization is not over stopping times. Instead, we find that the optimization is over the hazard rates of the driving Poisson processes.

Theorem. 4.3 *The solution of the system (60) is given by the value function*

$$\begin{aligned}P(S,L,t) &= \sup_{z\in\mathcal{Y}}\inf_{y\in\mathcal{Y}}\mathbb{E}_t^{\hat{\mathbb{Q}}}\left[e^{-\int_t^T(r+\hat\kappa(s))\,ds}\left\{h(L(T),S(T))\right.\right.\\ &\left.\left.+\frac{1}{\hat\alpha}\int_t^T e^{\int_u^T\hat\kappa(s)\,ds}\left(\frac{\hat\lambda(u)}{y(u)}\hat\beta(y(u))-\hat\kappa(u)\,\hat\beta(z(u))\right)du\right\}\right]\end{aligned}\qquad(68)$$

where $\mathcal{Y}$ is the set of non-negative $\mathcal{F}_t$-adapted processes, the loss process

$$L(t)=\sum_{n=1}^{\hat N(t)}g(S(t_i),t_i)\qquad(69)$$

and t_i are the arrival times of the doubly-stochastic Poisson process $\hat N(t)$. In the measure $\hat{\mathbb{Q}}$, the $\mathcal{F}_t$-adapted hazard rate process for $\hat N(t)$ is

$$\hat\lambda(t)=y(t)\,\lambda(t)\,e^{\alpha(t)g(S(t),t)}\;,\qquad(70)$$

and $\hat\kappa(t)=z(t)\kappa(t)$. Finally, $S(t)$ satisfies the SDE:

$$dS(t)=r\,S(t)\,dt+\sigma\,S(t)\,d\hat X(t)\,,\qquad(71)$$

where $\{\hat X(t)\}_{0\le t\le T}$ is a $\hat{\mathbb{Q}}$-Wiener process.

Proof. Let $\beta(x)$ denote the non-linear term in (60), i.e.

$$\beta(x)=1-e^{-x}\,,\qquad(72)$$

and let $\hat\beta(y)$ denote its convex-dual so that

$$\hat\beta(y)=\max_x\left(\beta(x)-x\,y\right)=1-y+y\,\ln y\,.\qquad(73)$$

Clearly, $\hat{\beta}(y)$ is defined on $(0,\infty)$ and is non-negative on its domain of definition. Furthermore,

$$\beta(x) = \min_{y\geq 0}\left(\hat{\beta}(y) + y\,x\right) . \tag{74}$$

Rewriting the exponential terms in (60) in terms of their convex dual, we find that the PDE becomes linear in P:

$$\begin{cases} r\,P = & P_t + r\,S\,P_S + \frac{1}{2}\sigma^2\,S^2\,P_{SS} \\ & +\frac{\kappa(t)}{\alpha(t)}\,\max_{z(t)\geq 0}\left(-\hat{\beta}(z(t)) - z(t)\,\alpha(t)\,P(L,S,t))]\right) \\ & +\frac{\lambda(t)}{\alpha(t)}e^{\alpha(t)g(S,t)}\,\min_{y(t)\geq 0}\left(\hat{\beta}(y(t)) + y(t)\,\alpha(t)\Delta P(L,S,t)\right)\,, \\ P(L,S,T) = & h(L,S)\,. \end{cases} \tag{75}$$

Through the usual dynamic programming principle, we find that the value function (68) satisfies the above HJB equation. □

We have shown that the pricing problem reduces to simultaneously finding the activity rate which minimizes and the interest rate that maximizes the Black-Scholes price of the reinsurance contract, subject to a penalty term – which is itself a function of the activity rate and interest rates. It is useful to illustrate how the risk-neutral result of the previous subsection is recovered. In the limit in which $\hat{\alpha}\to 0^+$, the penalty term increases to infinity and the process y which minimizes (68) is clearly the one in which $\hat{\beta}(y(u)) = 0$ for all $u\in[t,T]$. Similarly, the process z which minimizes (68) satisfies $\hat{\beta}(z(u)) = 0$. This is achieved when $y(u) = z(u) = 1$. The optimal hazard rate is then equal to its real world value $\hat{\lambda}(t) = \lambda(t)$ and the rate $\hat{\kappa}(t) = \kappa(t)$ implying an interest rate of $r+\kappa(t)$. The price therefore reduces to (64).

4.3. Numerical examples

In the absence of explicit solutions, we now demonstrate how the pricing PDE can be used, nonetheless, to obtain the value of reinsurance contracts through a simple implicit-explicit finite-difference scheme. Since we are not concerned with proving that the scheme converges in a wide class of scenarios, we take a practitioner's viewpoint and apply the scheme to situations in which the loss function and reinsurance contract itself are both bounded and asymptotically constant. To this end, it is convenient to rewrite the problem using the log of the forward-price process $z(t)\equiv \ln S(t) + r(T-t)$. Also, it is appropriate to scale the price function by the risk-aversion parameter and the discount factor by introducing the function

$$\overline{P}(L,z,t) \equiv \alpha(t)\,P(L,e^{z-r(T-t)},t)\,. \tag{76}$$

With these substitutions, the pricing PDE (60) becomes

$$\begin{cases} 0 & = \overline{P}_t - \frac{1}{2}\sigma^2\,\overline{P}_z + \frac{1}{2}\sigma^2\,\overline{P}_{zz} \\ & \quad -\kappa(t)\left(1-e^{-\overline{P}(L,z,t)}\right) + \overline{\lambda}(z,t)\,\left(1-e^{-(\overline{P}(L+\overline{g}(z,t),z,t)-\overline{P}(L,z,t))}\right)\,, \\ \overline{P}(L,z,T) & = \hat{\alpha}\,h(L,e^z)\,, \end{cases} \tag{77}$$

where $\overline{g}(z,t) = g(e^{z-r(T-t)},t)$ and $\overline{\lambda}(z,t) = \lambda(t)\,e^{\alpha(t)\overline{g}(z)}$. Now, we introduce a $M_L\times M_z\times N$ grid for the (L,z,t) plane with step sizes of $(\Delta L,\Delta z,\Delta t)$ so that

$$L_j = j\Delta L\,,\quad z_k = z_{min} + k\Delta L\,,\quad t_n = n\Delta t\,. \tag{78}$$

In addition, let $\overline{P}^n_{j,k} \equiv \overline{P}(L_j, z_k, t_n)$, $\overline{\lambda}^n_k \equiv \overline{\lambda}(z_k, t_n)$ and $\kappa^n \equiv \kappa(t_n)$. Treating the non-linear terms explicitly, and the derivatives in z via a θ implicit-explicit scheme, reduces (77) to the system of linear equations:

$$\begin{cases} (\theta-1)\,a\,\overline{P}^{n-1}_{j,k-1} + b\,\overline{P}^{n-1}_{j,k} + (\theta-1)\,c\,\overline{P}^{n-1}_{j,k+1} = \theta\,a\,\overline{P}^{n}_{j,k-1} + d\,\overline{P}^{n}_{j,k} + \theta\,c\,\overline{P}^{n}_{j,k+1} \\ \qquad\qquad -\kappa^n\left(1-e^{-\overline{P}^n_{j,k}}\right) + \overline{\lambda}^n_k\left(1-e^{-\Delta\overline{P}^n_{j,k}}\right), \\ \overline{P}^N_{j,k} = h(L_j, e^{z_k}), \end{cases} \tag{79}$$

where,

$$a = \tfrac{1}{2}(1+\tfrac{1}{2}\Delta z)\widetilde{\sigma}^2 \tag{80}$$

$$b = 1-\tfrac{1}{2}(\theta-1)\widetilde{\sigma}^2 \tag{81}$$

$$c = \tfrac{1}{2}(1-\tfrac{1}{2}\Delta z)\widetilde{\sigma}^2 \tag{82}$$

$$d = 1-\tfrac{1}{2}\theta\widetilde{\sigma}^2 \tag{83}$$

and,

$$\widetilde{\sigma}^2 \equiv \frac{\Delta z}{\Delta t^2}\,\sigma^2\,. \tag{84}$$

Finally, the nonlocal term $\Delta\overline{P}^n_{j,k}$ is treated in the approximation by linearly interpolating between neighboring points. Specifically, define $j'^n_k \equiv (L_j + g(z_k,t_n))/\Delta L$ (the number of non-integer steps to the loss point after a claim arrives) and $\gamma \equiv j'^n_k - \lfloor j'^n_k \rfloor$ (the fraction of the grid on which this new loss lies) then,

$$\Delta P^n_{j,k} \equiv \gamma\,P^n_{\lceil j'^n_k\rceil,k} + (1-\gamma)\,P^n_{\lfloor j'^n_k\rfloor,k} - P^n_{j,k}\,. \tag{85}$$

To complete the description of the finite difference scheme, we need to impose appropriate boundary conditions. For the numerical examples in this section, we assume that claim sizes are as described in §3.3.3. Since such claims are asymptotically constant as S approaches zero and infinity, the price of the reinsurance must approach the price of the constant loss case in these limits. This boundary condition may be implemented in two ways: (i) solving the constant loss case via an ODE solver, or (ii) imposing the condition that first derivative vanishes on the boundaries $\overline{P}_S(L, z_{min}, t) = \overline{P}_S(L, z_{max}, t) = 0$. Since the nonlocal term only requires upwind differences, only the boundary condition along the upper boundary for losses $L = M_L\,\Delta L$ needs specifications. For this, we assume that the reinsurance contracts are asymptotically constant in losses, and therefore impose the condition that $\overline{P}_L(L_{max}, z, t) = 0$; furthermore, any nonlocal term involving points above L_{max} are pulled back to L_{max}.

To illustrate the effects of the insurer's risk aversion level on the pricing results, we focus on three prototypical reinsurance payoff functions:

$$h_1(L,S) = \min(M, \max(0, L-m))\,, \tag{86}$$

$$h_2(L,S) = \mathbb{I}(S > S^*)\,\min(M, \max(0, L-m)) \quad \text{and} \tag{87}$$

$$h_3(L,S) = \mathbb{I}(S \le S^*)\,\min(M, \max(0, L-m))\,. \tag{88}$$

The first reinsurance payoff function h_1 corresponds to a stop-loss reinsurance contract with payments starting at losses of m and attaining a maximum of M. This reinsurance contract makes payments that are independent of the risky asset's price at maturity; however, because the loss sizes are linked to the equity value, the value of the contract at initiation will depend on the spot price of the risky asset. The second reinsurance payoff function h_2 corresponds to a double-trigger reinsurance contract in which a stop-loss payment is made if the risky asset's price rises above a critical value S^*. Such a contract could be attractive if the insurer is adverse to the potential of making larger payments when the index is doing well. As a counterpoint, the third payoff function h_3 corresponds to a double-trigger reinsurance contract in which a stop-loss payment is made if the risky asset's price drops below a critical value S^*. An insurer is willing to purchase this contract if they are adverse to falling profits in their equity investment when the index performs poorly.

In Figure 3, we illustrate how the price of the three reinsurance contracts depend on the prevailing spot price for several levels of risk aversion $\hat{\alpha}$. As expected, the price increases as the insurer becomes more risk averse. Furthermore, for any given risk-aversion level, the price of either double-trigger stop-loss contract is lower than the pure stop-loss contract. This too is expected since the double-trigger contract pays nothing if the risky asset's price is below the trigger level at maturity. Finally, in the stop-loss contract's price approaches the price of double trigger contract with an upper trigger for large spot prices, while it approaches the price of the double-trigger contract with a lower trigger for small spot prices.

5. Conclusions

In this paper, we obtained the premium an insurer requires if she takes on the risk of equity-linked losses. To do so, we employed the principle of equivalent utility with exponential utility to value the contract. Even though the insurer is risk-averse, we demonstrated that the premium is obtained by computing a risk-neutral expectation of an exponentially weighted average of the claim sizes. In the limit in which the insurer becomes risk-neutral this expectation reduces to the expected loss per unit time under the unique minimal martingale measure. Furthermore, we examined the price that an insurer, already exposed to the equity-linked risks, would be willing to pay for a reinsurance contract paying a function of both the equity value and total losses. The potential default of the counter-party was simultaneously accounted for. The general non-linear PDE that arises from the associated HJB equations was not solved in complete generality. However, we were able to rewrite the non-linear PDE in terms of a dual linear optimization problem. This allowed us to provide a probabilistic interpretation of the pricing problem for the reinsurance contract: The price in the dual representation, is a minimum of the risk-neutral price plus a penalty term, where the optimization is over a stochastic activity rate for the Poisson process driving the claim arrival times and stochastic interest rates. In the limit of a risk-neutral insurer, we demonstrated that the price reduces to an expectation of the reinsurance payoff over a risk-neutral measure in which the distribution of losses are identical to the real-world distribution while interest rates are default adjusted.

There are several avenues that are open for further exploration. For example, it would be

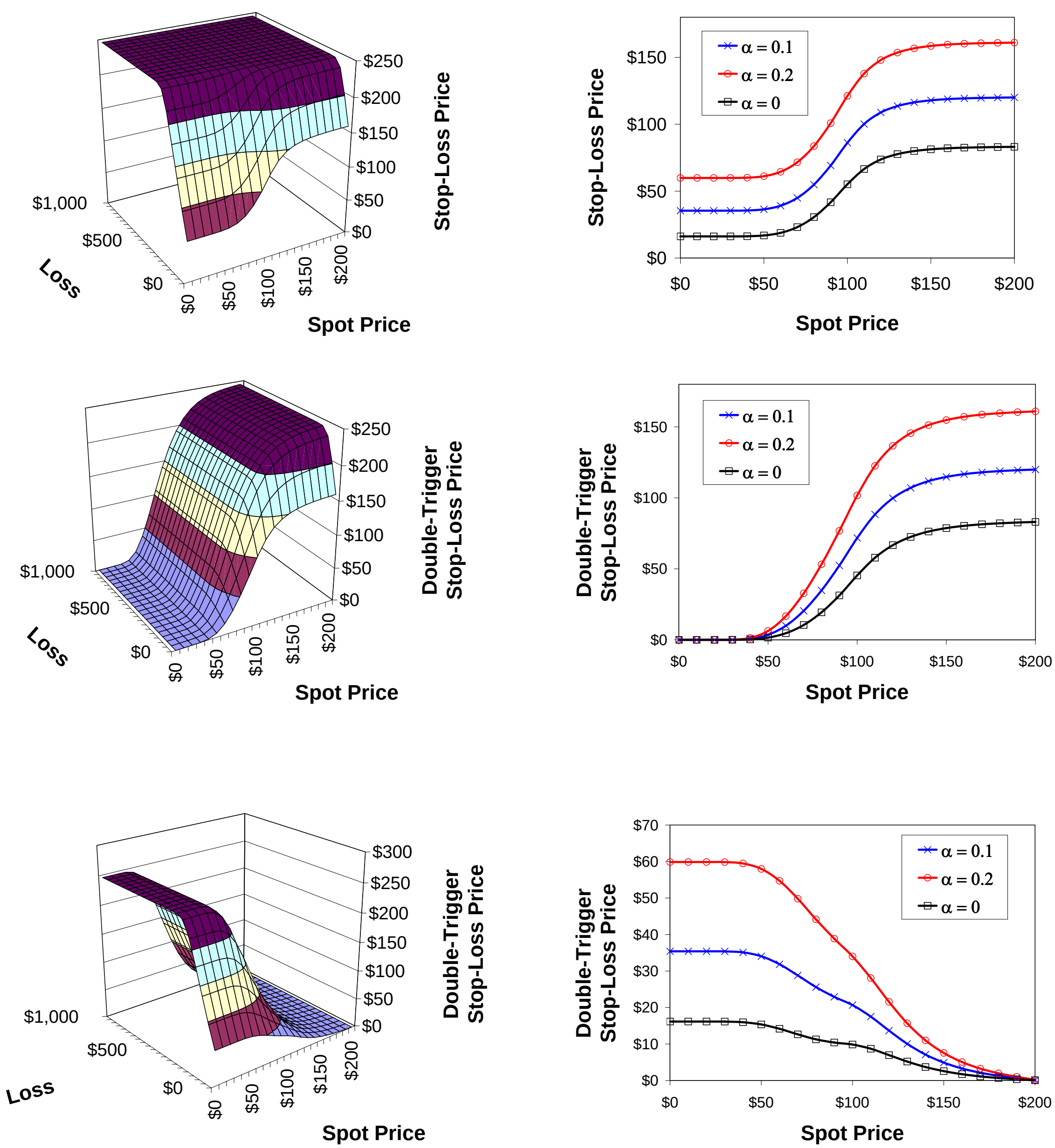

Figure 3. The indifference price for the three reinsurance contracts (86), (87) and (88)with losses described in §3.3.3. The model parameters are those used in Figure 1. In the left panels, the risk-aversion parameter is set to $\alpha = 0.2$. In all experiments, we used 1000 time steps and a 1000×1000 grid in the (L, z) plane, with $z_{min} = -10$, $z_{max} = 10$ and $L_{max} = 2000$.

interesting to obtain the distribution of the ruin times for insurers facing these equity-linked risks. A closed form result for general loss functions $g(S, t)$ is not likely. We suspect, however, that in cases when g is a piecewise linear function of the log-stock price, a semi-explicit form might be available. Extending the results to incorporate a finite number of claim sources is also interesting. Likely, a $\frac{1}{N}$ expansion where N is the number of potential claims will lead to interesting correction terms. Another exploration could involve ruin-related problems: such as the optimal consumption problem for the insurer where the value function truncates at the time of ruin. This is similar to the questions Young and Zariphopoulou (2002) addressed in the context of fixed loss sizes. Extending their results to the case of equity-linked losses would be quite interesting. The Gerber and Shiu (1998) penalty function is another problem related to the time of ruin, and although it is not explicitly connected to question of indifference pricing, it would also be interesting to investigate its equity-linked extensions.

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