

Value a call on an asset S , strike K , mat T .

$$V_0 = e^{-rT} \mathbb{E}^Q [(S_T - K)_+]$$

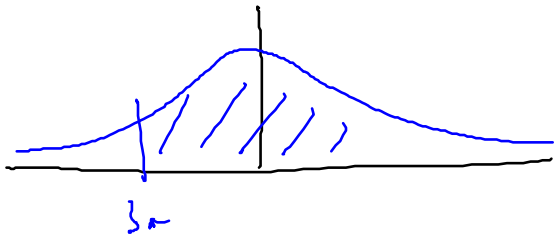
$$S_T \stackrel{d}{=} S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}z}, \quad z \sim N(0,1)$$

$$\mathbb{E}^Q [(S_T - K) \mathbb{1}_{S_T > K}]$$

$\mathbb{1}_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$

$$= \mathbb{E}^Q [S_T \mathbb{1}_{S_T > K}] - K \mathbb{E}^Q [\mathbb{1}_{S_T > K}]$$

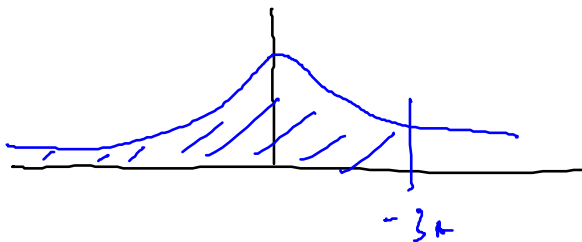
$$\begin{aligned} &\hookrightarrow \mathbb{Q}(S_T > K) \\ &= \mathbb{Q}\left(z > - \frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \end{aligned}$$



\hookrightarrow

$$= \Phi\left(\frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

\hookrightarrow std. normal cdf



$$\mathbb{E}^Q [S_T \mathbb{1}_{S_T > K}] = \int_{-\infty}^{\infty} S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}z} \mathbb{1}_{z > z^*} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz$$

$$= S_0 e^{(r - \frac{1}{2}\sigma^2)T} \int_{z^*}^{\infty} e^{\sigma\sqrt{T}z - \frac{1}{2}z^2} \frac{dz}{\sqrt{2\pi}}$$

$\hookrightarrow I$

$$I = \int_{z^*}^{\infty} e^{-\frac{1}{2}(\underbrace{z - \sigma\sqrt{T}}_{z'})^2 + \frac{1}{2}\sigma^2 T} \frac{dz}{\sqrt{2\pi}}$$

$$\begin{aligned}
&= e^{\frac{1}{2}\sigma^2 T} \int_{S^* - \sigma\sqrt{T}}^{\infty} e^{-\frac{1}{2}(z')^2} \frac{dz'}{\sqrt{2\pi}} \\
&= e^{\frac{1}{2}\sigma^2 T} \Phi(\sigma\sqrt{T} - z^*) \\
&= e^{\frac{1}{2}\sigma^2 T} \Phi\left(\frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)
\end{aligned}$$

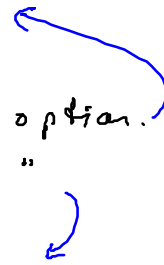
$$V_0^c = S_0 \Phi(d_+) - K e^{-rT} \Phi(d_-)$$

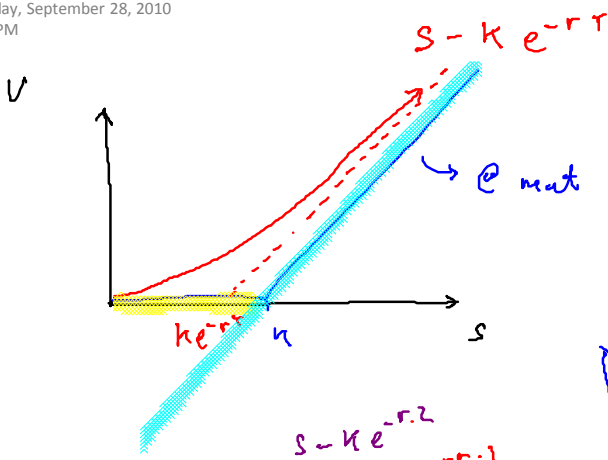
$$d_{\pm} = \frac{\ln(S_0/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Black-Scholes Formula For a call option.

$$V_0^p = K e^{-rT} \Phi(-d_-) - S_0 \Phi(-d_+)$$

put

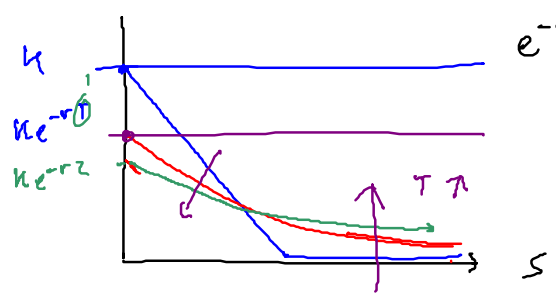
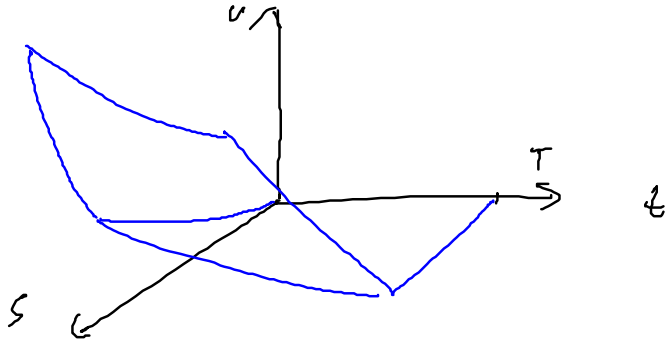
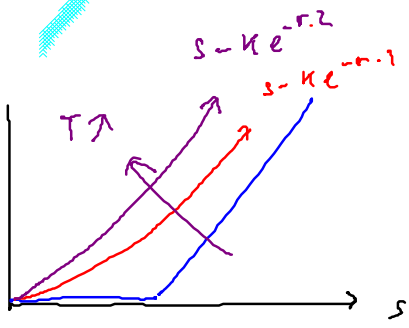




$$Q^c \geq S_T - K$$

$$\downarrow \quad \downarrow$$

$$V^c \geq S_0 - Ke^{-rT}$$

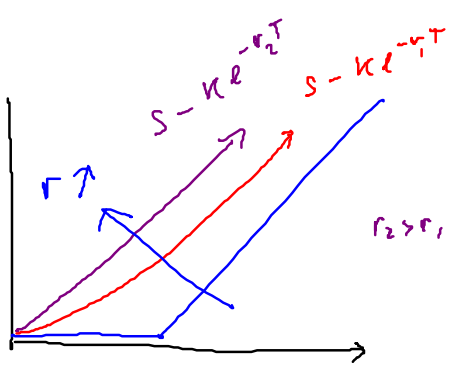


$$e^{-rT} \mathbb{E}^Q [(K - S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sigma T Z})_+]$$

$$Q^P \leq K$$

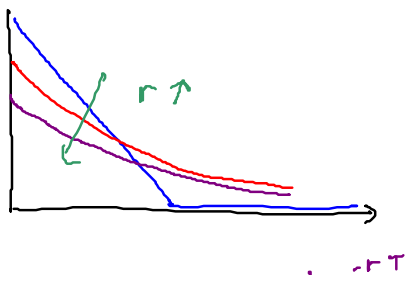
$$\downarrow$$

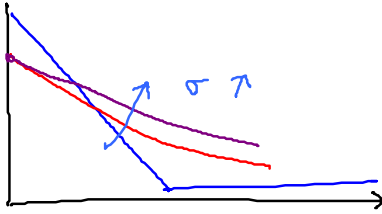
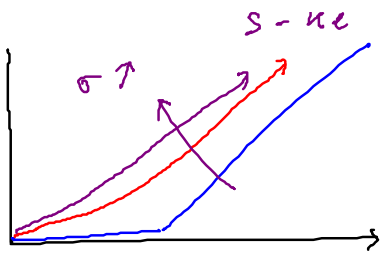
$$V^P \leq Ke^{-rT}$$



$$V_c = e^{-rT} \mathbb{E}^Q [(S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sigma T Z} - K)_+]$$

$$= \mathbb{E}^Q [(S_0 e^{-\frac{1}{2}\sigma^2 T + \sigma\sigma T Z} - Ke^{-rT})_+]$$

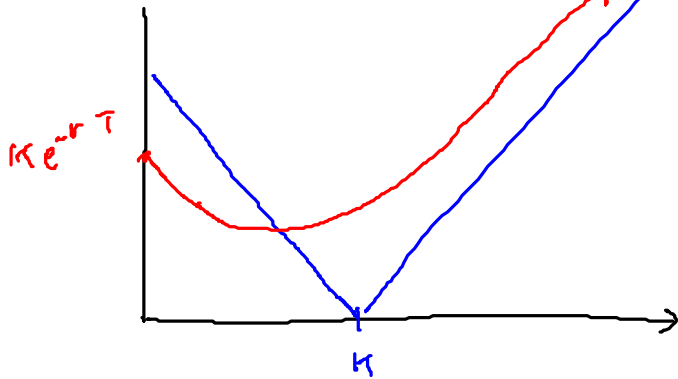




$$S - Ke^{-rt}$$

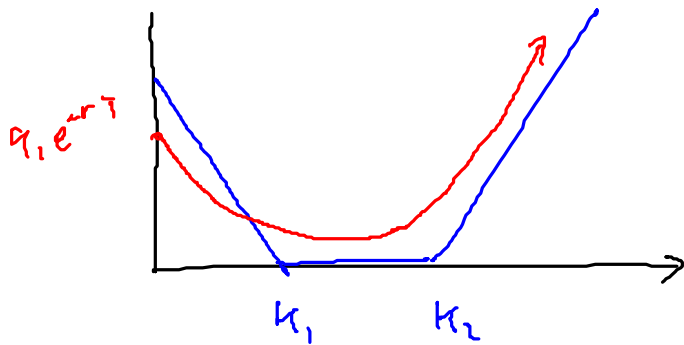
straddle

put K
call K

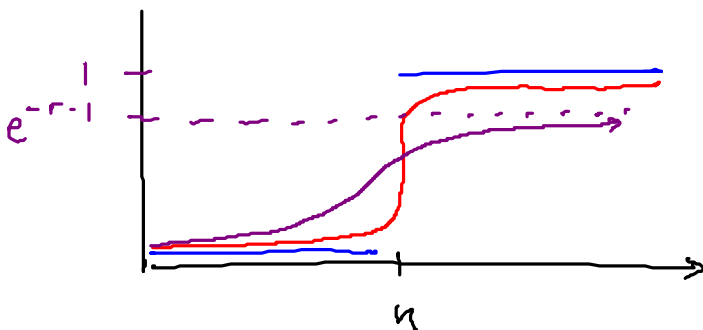
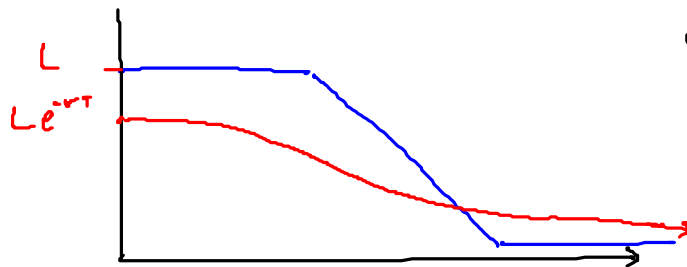


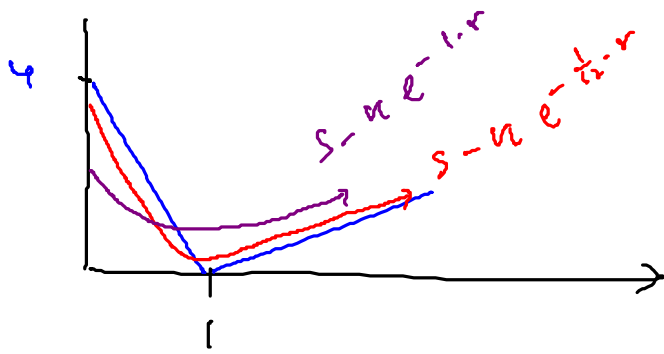
$$S - K_2 e^{-rt}$$

strangle



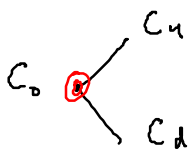
bear spread





American styled claims: can exercise at any time.

in principle can have black-out periods.



① hold H_0 or ② exercise E_0

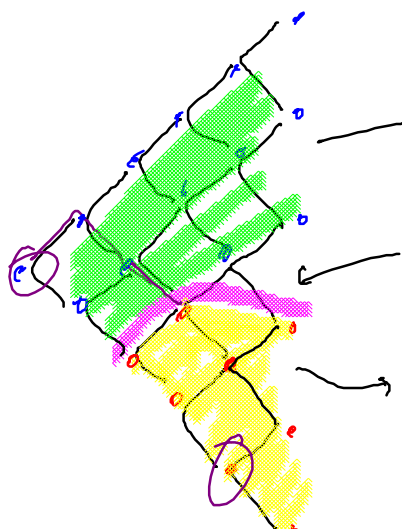
$$C_0 = \max(H_0, E_0)$$



(Part)

$$E_0 = (K - S_0)_+$$

$$H_0 = e^{-rt} (q C_u + (1-q) C_d)$$

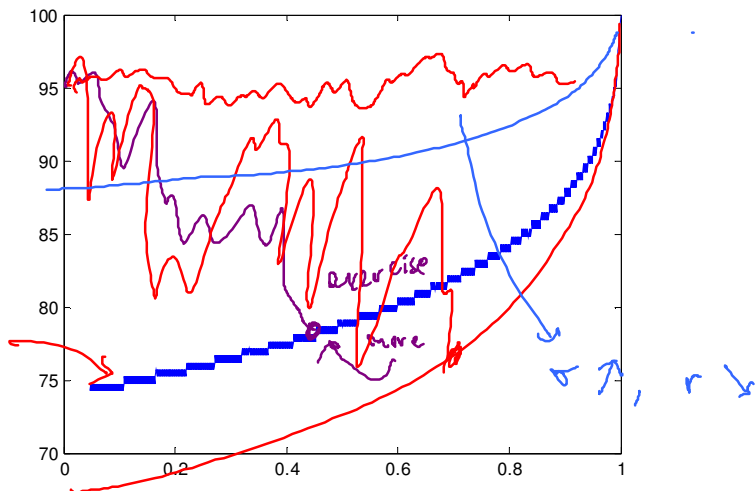
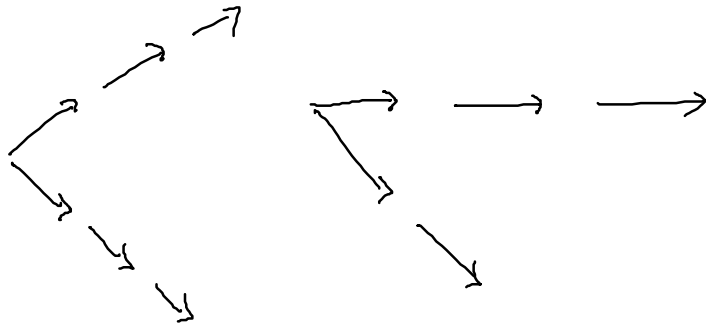
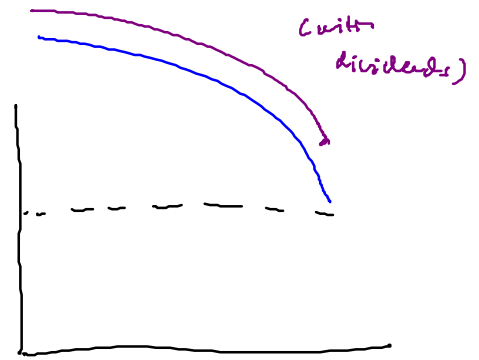
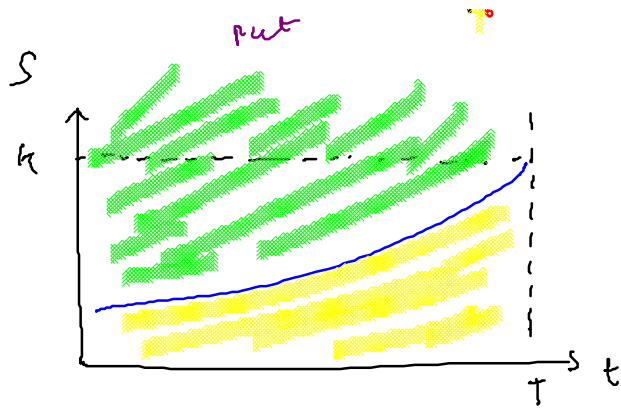


continuation region

optimal exercise curve

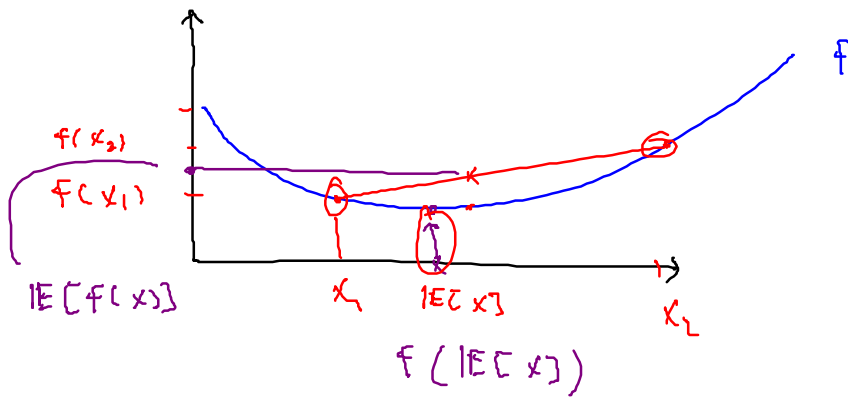
exercise region

call



$F(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ Convex

$$\mathbb{E}[F(X)] \geq F(\mathbb{E}[X])$$

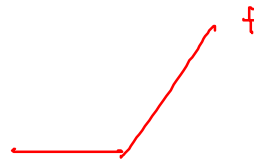


Q: show that it is never optimal to exercise a call option early. [no dividends]

$$\textcircled{1} \quad \mathbb{E}^Q [S_T] = S_0 e^{rT}$$

$$\textcircled{2} \quad \mathbb{E}[f(x)] \geq f(\mathbb{E}[x])$$

↑? ↓?



$$H_0 = e^{-rT} \mathbb{E}^Q [(S_T - K)_+] \quad f(x) = (x - K)_+ \text{ is convex.}$$

$$\therefore \mathbb{E}^Q [f(S_T)] \geq f(\mathbb{E}^Q[S_T]) = f(S_0 e^{rT})$$

↳ Jensen's inequality

$$= (S_0 e^{rT} - K)_+$$

$$\Rightarrow H_0 \geq e^{-rT} (S_0 e^{rT} - K)_+ = (S_0 - Ke^{-rT})_+$$

$$\geq (S_0 - K)_+ = E_0$$

\therefore never optimal to exercise.