ACT 460 / STA 2502 Stochastic Methods for Actuarial Science - Problem Set #4
Not for grading.

1. Consider each of the following options:

   (a) digital call struck at 100

   (b) digital put struck at 100

   (c) put struck at 100

   (d) call struck at 100

   (e) strangle struck at 100

   (f) straddle with $K_1 = 95$, $K_2 = 115$

   (g) bull spread with $K_1 = 95$, $K_2 = 115$

   Assuming the Black-Scholes model, use Excel to plot the price, the delta, and the gamma versus spot level for each option using the following sets of parameters (put each parameter set on a single plot):

   (a) $T = \{\frac{1}{12}, \frac{1}{4}, \frac{1}{2}\}$; $\sigma = 20\%$; $r = 5\%$; $\delta = 3\%$

   (b) $T = \frac{1}{4}$; $\sigma = \{10\%, 20\%, 30\%\}$; $r = 5\%$; $\delta = 3\%$

   (c) $T = \frac{1}{4}$; $\sigma = 20\%$; $r = \{0\%, 5\%, 10\%\}$; $\delta = 3\%$

   (d) $T = \frac{1}{4}$; $\sigma = 20\%$; $r = 5\%$; $\delta = \{0\%, 3\%, 6\%\}$

2. Derive the delta and gamma for a digital put and digital call option using the Black-Scholes model.

3. Using the Black-Scholes model, determine the price, the delta and the gamma at time t of the following European options with payoffs at time $T > t$:

   (a) A forward-start digital call option, which pays 1 at T if the asset price at maturity is above a percentage α of the asset price at time U (where $t < U < T$). That is, $\varphi = \mathbb{I}(S_T > \alpha S_U)$.

   (b) A forward-start asset-or-nothing option which pays the asset at T if the asset price at maturity is above a percentage α of the asset price at time U (where $t < U < T$). That is, $\varphi = S_T\,\mathbb{I}(S_T > \alpha S_U)$.

   (c) A call option (maturing at V) on a forward-start asset-or-nothing option. The embedded forward-start asset-or-nothing option pays the asset at $T > V$ if the asset price at T is above a percentage α of the asset price at time U (where $V < U < T$). The strike of the call option is K.

4. Suppose that the price of a stock is modeled as follows:

$$\frac{dS_t}{S_t} = \mu_t \, dt + \sigma_t \, dW_t$$

where μ_t and σ_t are functions only of time and where W_t is a $\mathbb{P}$-Wiener process. Furthermore, assume that the risk-free interest rate r_t is function only of time. Determine the price, the delta and the gamma for each of the following options:

   (a) call option maturing at T strike of K.

   (b) forward starting put option with strike set to αS_U at time U and maturing at T.

5. Suppose that interest rates follow the Ho-Lee model:

$$dr_t = \alpha_t \, dt + \sigma \, dW_t$$

where α_t is a deterministic function of time and W_t is a $\mathbb{Q}$-Wiener process. Determine each of the following:

   (a) Bond price at time t of maturity T.

   (b) The SDE which the bond price satisfies in terms of W_t.

   (c) The choice of α_t which makes the model prices equal the market prices $P_t^*(T)$.

6. Use the HedgeSim.xls file to investigate various aspects of discrete hedging on a call option which matures in 60 trading days (60/252 of a year) struck at 100 on an underlier with spot of 100, realized vol of 20% and realized return of 10%. Assume the risk-free rate is 5%. For each of the following scenarios obtain the PnL through Delta hedging using 2, 10 and 60 rebalances.

   - Assume the market trades options at a vol of 20% and you rebalance the portfolio 12 times using a hedging vol of 20%.
   - Assume the market trades options at a vol of 25% and you rebalance the portfolio 12 times using a hedging vol of 20%.
   - Assume the market trades options at a vol of 25% and you rebalance the portfolio 12 times using a hedging vol of 25%.
   - Assume the market trades options at a vol of 25% and you rebalance the portfolio 12 times using a hedging vol of 22.5%.
   - Comments?
   - Repeat the above with Delta-Gamma hedging using a call option struck at 100 but maturing at in 80 trading days to match gamma.