ACT 460 / STA 2502 Stochastic Methods for Actuarial Science - Problem Set #1 due Tuesday, Sept 25: Hand in only questions marked with (**).

1. Using a CRR tree, with $S = 100$, $\sigma = 50\%$, $\delta = 3\%$, $r = 5\%$ and $\Delta t = \frac{1}{12}$, determine the value and hedging strategy for each of the following 3-month European options:

   (a) digital call struck at 100

   (b) digital put struck at 100

   (c) put struck at 100

   (d) call struck at 100

   (e) strangle struck at 100

   (f) straddle with $K_1 = 95$, $K_2 = 115$

   (g) bull spread with $K_1 = 95$, $K_2 = 115$

2. Assuming the Black-Scholes model, use Excel to plot the price versus spot level for each option in Q1 using the following sets of parameters (put each parameter set on a single plot):

   (a) $T = \{\frac{1}{4}, \frac{1}{2}, 1\}$; $\sigma = 20\%$; $r = 5\%$; $\delta = 3\%$

   (b) $T = 1$; $\sigma = \{10\%, 20\%, 30\%\}$; $r = 5\%$; $\delta = 3\%$

   (c) $T = 1$; $\sigma = 20\%$; $r = \{0\%, 5\%, 10\%\}$; $\delta = 3\%$

   (d) $T = 1$; $\sigma = 20\%$; $r = 5\%$; $\delta = \{0\%, 3\%, 6\%\}$

3. Assuming the Black-Scholes model, use Excel to run a Monte Carlo simulation and obtain the present value profit and loss histogram of the options in Q1 for each of the parameter sets in Q2 assuming that $S = 100$ and $\mu = 10\%$. Use $50,000$ simulations to obtain the PnL.

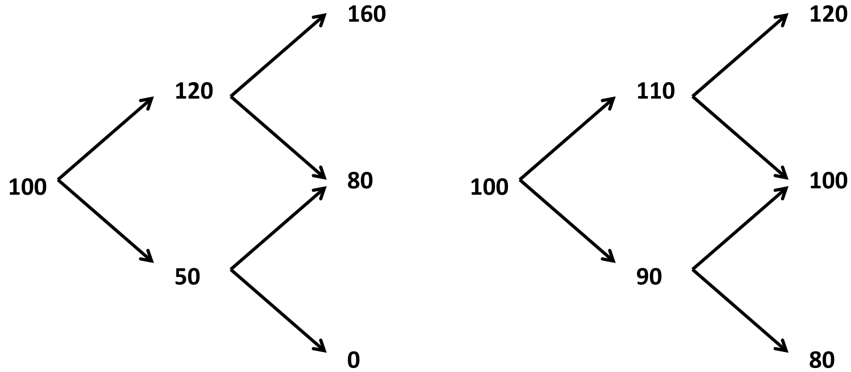
4. Let $\{t_j : j = 0, \dots, m\}$ be an ordered series of times $t_0 = 0 < t_1 < t_2 < \dots < t_m = T$, and let $\{X_i : i = 1, \dots, m\}$ denote a set of independent normal random variables with means of $\mu\,(t_i - t_{i-1})$ variances of $\sigma^2\,(t_i - t_{i-1})$. Suppose that an asset's price at time t_i are modeled by exponentiation of these r.v.s:

$$S(t_i) = S(t_{i-1})\, e^{X_i}\,. \qquad (1)$$

   (a) Define $\overline{S}(n)$ as the geometric average of the asset's price over the first n ordered times ($n \le m$). That is, $\overline{S}(n) := \left(\prod_{j=1}^{n} S(t_j)\right)^{1/n}$. What is the distribution of $\overline{S}(n)$?

   (b) Determine the probability that $\overline{S}(n) > K$ given that $\overline{S}(m) > K$, where $m < n$. Write the result in terms of the normal distribution function $\Phi(a) := \mathbb{P}(N_1 < a)$ and bivariate normal distribution function $\Phi(a, b; \rho) := \mathbb{P}(N_1 < a, N_2 < b)$ where N_1 and N_2 are standard normal random variables with correlation ρ.

5. The following two assets are being actively traded in a two-period binomial market economy. Asset A behaves like a stock which may default, while asset B behaves "normally".



(a) Asset A (b) Asset B

(a) **[5]**** Determine all relevant risk-neutral probabilities and short rates of interest.

(b) **[5]**** Using risk-neutral valuation, compute the price and replication strategy for a two-period American put option on asset A struck at 90.

6. Real options are pervasive in industry and appear in many different forms, particularly in the manufacturing sector. They are often (wrongly) valued by simply computing the expected value of the claim's payoff and discounting the expected payoff back to the start. Using this näive method, compute the price of real options which pay

(a) S_T

(b) $1/S_T$

(c) $\mathbb{I}(S_T > K)$

(d) $S_T\,\mathbb{I}(S_T > K)$

(e) $(S_T - K)_+$

(f) $(K - S_T)_+$

(g) **[5] (**)** $(S_T - K)_+\mathbb{I}(S_T > \alpha\, K)$ where $\alpha > 1$

at the maturity date T, assuming that $S_T = S_0\, e^X$ where $X \sim \mathcal{N}\left((\mu - \frac{1}{2}\sigma^2)T,\, \sigma^2\, T\right)$ and the continuous risk-free rate is r.

7. **Grad students hand in this question as well.** You are modeling the evolution of two correlated stocks as follows:

$$S_1(i\,\Delta t) = S_1((i-1)\,\Delta t)\, e^{\alpha_1\, x_i} \qquad S_2(i\,\Delta t) = S_2((i-1)\,\Delta t)\, e^{\alpha_2\, y_i} \tag{2}$$

where $(x_1, y_1), \ldots, (x_n, y_n)$ are pairwise i.i.d Bernoulli random variables with the following real-world probabilities:

$$\mathbb{P}(x_i = +1,\ y_i = +1) = p_1 \qquad \mathbb{P}(x_i = +1,\ y_i = -1) = p_2 \tag{3}$$
$$\mathbb{P}(x_i = -1,\ y_i = +1) = p_3 \qquad \mathbb{P}(x_i = -1,\ y_i = -1) = p_4 \tag{4}$$

You wish to ensure that in the limit in which $n \to \infty$ with $\Delta t = T/n$ (T fixed and finite), the joint distribution of $S_1(T)$ and $S_2(T)$ is a joint log-normal with the following properties

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[S_1(T)\right] &= e^{\mu_1 T} S_1(0) \\
\mathrm{Var}^{\mathbb{P}}\left[\ln(S_1(T)/S_1(0))\right] &= \sigma_1^2 T \\
\mathbb{E}^{\mathbb{P}}\left[S_2(T)\right] &= e^{\mu_2 T} S_2(0) \\
\mathrm{Var}^{\mathbb{P}}\left[\ln(S_2(T)/S_2(0))\right] &= \sigma_2^2 T \\
\mathrm{Corr}(\ln(S_1(T)/S_1(0)), \ln(S_2(T)/S_2(0))) &= \rho
\end{aligned}$$

Determine the probabilities $\{p_j : j = 1, 2, 3, 4\}$ and $\alpha_{1,2}$ in terms of the model parameters and Δt to lowest order in Δt.

8. **[5]** ** Assuming that interest rates are model as follows:

$$r_{n\Delta t} = r_{(n-1)\Delta t}\, e^{\sigma\sqrt{\Delta t}\, X_n}$$

where $X_1, X_2, \ldots$ are iid Bernoulli random variables with $\mathbb{Q}(X_j = +1) = \frac{1}{2}$ and $\mathbb{Q}(X_j = -1) = \frac{1}{2}$. Assume that $r_0 = 5\%$, $\sigma = 10\%$ and $\Delta t = \frac{1}{2}$. Use Excel to:

(a) Determine the price of a 4-year zero coupon bond on a notional of 1 million.

(b) Determine the price of a call option maturing in 2-years on the bond in part(ii) with a strike equal to 0.95 million.

(c) Determine the replication strategy for the call option in part (ii) using the 4-year zero bond and the 3-year zero bond as replication instruments (both on notionals of 1 million).