Week # 1

From the text.

P16

1. If Axioms 1 + 4 holds then

$a \le X \le b \Rightarrow b - X \ge 0$ + $X - a \ge 0 \Rightarrow b \ge E(X)$ + $E(X) \ge a$

$\Rightarrow a \le E(X) \le b$

If $a \le X \le b \Rightarrow a \le E(X) \le b$, $\forall a, b$ then clearly $X \ge 0 \Rightarrow E(X) \ge 0$ (take $a = 0, b = \infty$) + setting $X = 1$ yields $E(1) = 1$ (i.e take $a = b = 1$)

3. $X \le |X|$ + $-X \le |X|$ so that $E(X) \le E(|X|)$ and $-E(X) \le E(|X|)$ or $-E(|X|) \le E(X)$. Since $|y| \le c \Leftrightarrow \underbrace{-c \le y \le c}_{-c \le y \text{ and } y \le c}$ we conclude $|E(X)| \le E(|X|)$

4. $|a+b| \le |a| + |b|$, $\forall a, b$ so that

$|X_1 + X_2| \le |X_1| + |X_2| \Rightarrow E(|X_1 + X_2|) \le E[|X_1| + |X_2|]$

$= E(|X_1|) + E(|X_2|)$

5. $|X_n - X| \le Y_n \Rightarrow E(|X_n - X|) \le E(Y_n)$

Hence $0 \le \lim E(|X_n - X|) \le \lim E(Y_n) = 0$ so that

$\lim E(|X_n - X|) = 0$

Now $|E(X_n - X)| \le E(|X_n - X|) \to 0$ + so $|E(X_n) - E(X)| \to 0$

$\Rightarrow E(X_n) \to E(X)$

#3 - p21

3. $X(\omega) = 1, \forall \omega \Rightarrow \frac{1}{2D}\int_{-D}^{D} X(\omega)d\omega = 1$ so $E(1) = 1$

$X(\omega) \geq 0, \forall \omega \Rightarrow \int_{-D}^{D} X(\omega)d\omega \geq 0 \Rightarrow \lim \frac{1}{2D}\int_{-D}^{D} X(\omega)d\omega \geq 0$

(if the lim exists)

$$\int_{-D}^{D} X(\omega) + Y(\omega)\,d\omega = \int_{-D}^{D} X(\omega)\,d\omega + \int_{-D}^{D} Y(\omega)\,d\omega$$

$$\Rightarrow E(X+Y) = \lim_{D\to\infty}\left[\frac{1}{2D}\int_{-D}^{D} X(\omega)d\omega + \frac{1}{2D}\int_{-D}^{D} Y(\omega)\,d\omega\right]$$

$$= \lim_{D\to\infty}\frac{1}{2D}\int_{-D}^{D} X(\omega)d\omega + \lim_{D\to\infty}\frac{1}{2D}\int_{-D}^{D} Y(\omega)d\omega$$

$$= E(X) + E(Y)$$

$$E(cX) = \lim_{D\to\infty}\frac{1}{2D}\int_{-D}^{D} c\,X(\omega)\,d\omega$$

$$= c\lim_{D\to\infty}\frac{1}{2D}\int_{-D}^{D} X(\omega)\,d\omega = c\,E(X)$$

$$\left.\begin{aligned} X_n(\omega) &= 1, \quad |\omega| \leq n \\ &= 0, \quad \text{ow} \end{aligned}\right\} \Rightarrow X_n \uparrow 1$$

But

$$E(X_n) = \lim_{D\to\infty}\frac{1}{2D}\int_{-n}^{n} 1\,d\omega = \lim_{D\to\infty}\frac{n}{D} = 0$$

so that $E(X_n) \not\to 1$

#4 - p35

3. $E(X) = 0\left(1-\frac{1}{m}\right) + m^{\frac{1}{4}}\left(\frac{1}{2m}\right) + \left(-m^{\frac{1}{4}}\right)\left(\frac{1}{2m}\right) = 0$

$$E(X^2) = \frac{m^{\frac{1}{2}}}{2m} + \frac{m^{\frac{1}{2}}}{2m} = \frac{1}{\sqrt{m}}$$

$$\Rightarrow Var(X) = E(X^2) - (E(X))^2 = \frac{1}{\sqrt{m}}$$

$$E(X^4) = 0\left(1-\frac{1}{m}\right) + m\left(\frac{1}{2m}\right) + m\left(\frac{1}{2m}\right) = 1$$

For large m $Var(X) \approx 0$ but $E(X^4) = 1$.

By letting, for any $c > 0$,
$P(X = 0) = 1 - \frac{1}{m}$, $P(X = (cm)^{1/4}) = \frac{1}{2m}$, $P(X = -(cm)^{\frac{1}{4}}) = \frac{1}{2m}$
we see $E(X^4) = c$ while $Var(X) = c/\sqrt{m}$. So no matter what c is just choose m large enough so that $Var(X) \approx 0$.

4. Since $I_{\{X \geq a\}} \leq \frac{H(X)}{H(a)}$ we get $E(I_{\{X \geq a\}}) \leq \frac{E(H(X))}{H(a)}$

$$\Rightarrow P(X \geq a) \leq E[H(X)] / H(a)$$

7. The variance matrix $\Sigma \geq 0$ so that $\det(\Sigma) \geq 0$ or $[cov(X, Y)]^2 \leq Var(X) Var(Y)$.
On the other hand, $\det(\Sigma) = 0$ (ie equality in (39))
$\Rightarrow \exists \, \underset{\sim}{c} \neq \underset{\sim}{0}$ st $Var\left(\underset{\sim}{c}'\begin{pmatrix} X \\ Y \end{pmatrix}\right) = 0$ or

$$c_1 X + c_2 Y \overset{\text{wp1}}{=} c_0$$