Identity for Statistics:
Calibration in Statistics

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What's up?

1. Statistics
2. Identity
3. Economist
4. Theorem
5. Priors
6. Linear parameters
7. Curved parameters
8. Curved models
9. Calibration!
1 Statistics: Why?

Why are we here? Statistics. MANY thanks to organizers!

Why am I here? Dumb numbers! The clock was set ticking long before I knew!

Came from another discipline?
- There was a book published on "probability!"
- Some mathematicians viewed it as profound & admired it.
- """""""it as wrong & condemned it.
- """""""were mystified.
- An aura of the 'religious' .... ?

The issues still pervade...: "Bayes 1763" despite attempts to hide.
Jeffreys: Theory of Probability 1939
Gave fundamental extensions to Bayes 1763
Geophysicist of highest stature!

vehemently against plate tectonics
continental drift

Deep things involved!
Beliefs ... re the earth!
... re how to think!

Toleration in statistics ... of contradictions?
... intolerable in science

What do sensible people think of "us"?
2 Identity

How do others see Statistics?
- You say you are a Statistician! ... Reactions: # ! & ... 
- Used to be that way for Mathematicians (-25BP) ... they fixed it!
- What can we do?

Not good enough to:

- Say we have a tool box ... then find most tools uncalibrated!
- Use our opinions ... to do the analyses!
- Promote contradictory tools ... then shrug: just "different way of thinking" ... What? Double talk!
- 'Shoot ourselves in the foot' publicly!
- Promote a religious view of our insights!
In praise of Bayes: upbeat... "proponent's views..."

"...value of a statistical method... apparent..."

"...results... easier to understand..."

"...clinical trials... faster"

"Not bad for an old dead white male"

Not much on: what the method is...

"...math rule... how you should change... beliefs..."

Some "f" bashing...

Quote: Larry W. ... "more a religion than a science"

"Evidently there is life in the old reverend yet."
4 The Theorem

***Good theorem***: Two inputs $f(x)$, $f(y|x)$
One output $f(x|y)$
In *controversible*!

***An application***: one of inputs is **missing**
So you **make one up**! Do some cosmetics...
then **Claim** the output is **valid**?

Even MAT 137 students can handle that!
Worse than "$\frac{a}{b}$"!

**Context**: There was a **value**... $\Theta$
"was a process $f(y;\Theta)$
"was a consequent value $y^0$
they are all in the past!

What do we know about the **value**... $\Theta$?
Do we assert **probability**? On what basis?
5 Types of Prior $\Pi(\theta)$

$$\Pi(\theta) \rightarrow \theta \quad f(y;\theta) \rightarrow y$$

1. **Real**  
   Known random source for the $\theta$

2. **Opinion**  
   Your betting instincts

3. **Convenient**  
   Hope for something useful

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**Does the Theorem apply to Application?**

- **Case 1.** Yes!
- **Case 3.** No! ... but things 'sort of work'

It is "Approximate confidence"

Laplace liked the answers!

... after all $y = x\beta + \epsilon$  
$\beta, \epsilon$ linear!
Linearity: $N(\theta; I)$ on $\mathbb{R}^2$; Contours of Interest $y_1(\theta) = \theta_1$  

(i) What probability says:

(ii) What Bayes says:
(i) What probability says: If \( \theta_1 = 7 \) then \( y \sim N(\frac{7}{2}, 1) \) \( y_1 \sim N \neq 1 \)

(ii) What Bayes says:
(i) What probability says: If $\theta_1 = 7$ then $y \sim N\left(1, \frac{3}{\theta_2}\right)$. $y_1 \sim N + 1$ for data $y_1^0 = 6$. ....

Prob left of data = $\Phi(-1) = \Phi\left(\frac{6-3}{1}\right)$

$= p$-value $= 16\%$

(ii) What Bayes says:
(i) What probability says: If $\theta_1 = 7$ then $y \sim N\{(\theta_1); I\}$ $y_1 \sim N \pm 1$ for data $y^0 = 6$...

Prob left of data = $\Phi(-1) = \Phi(\frac{6-7}{1})$

= $p$-value = 16%

(ii) What Bayes says: If $y^0 = 6$ then $\theta \sim N\{(\theta^0); I\}$ $\theta_1 \sim N 6 1$
(i) What probability says: If $\Theta_1 = \theta$ then $y \sim \mathcal{N}(\Theta_2; I)$ $y_1 \sim \mathcal{N}(\theta_2; I)$

For data $y_0 = 6$ ...

$\text{Prob left of data } = \Phi(-1) = \Phi\left(\frac{6-\theta}{I}\right)$

$= \text{p-value } = 16\%$

(ii) What Bayes says: If $y_0 = 6$

then $\Theta \sim \mathcal{N}(\theta_0; I)$ $\Theta \sim \mathcal{N}(6; 1)$

For para. value $\Theta_1 = \theta$ ...

'Prob' larger than $\Theta_1 = \theta$

$= s\text{-value } = 16\%$
(i) What probability says: If $\Theta_1 = 7$ then $y \sim N((\theta_2)'; I)$ $y_1 \sim N + 1$

For data $y_1 = 6$ ...

Prob left of data $= \Phi(-1) = \Phi(\frac{6-7}{1}) = \rho$-value $= 16\%$

(ii) What Bayes says: If $y^o = 6$

then $\Theta \sim N((6); I)$ $\Theta \sim N(6; 1)$

For para. value $\Theta_1 = 7$ ...

'Prob' larger than $\Theta_1 = 7$

= s-value $= 16\%$

They are equal! ... for any linear

$16\% = 16\%$ parameter

Bayes is "right on"!
Bayes "works"!
Bayes is easy, exact!

BUT...
What about curved parameters?

(i) What probability says: If \( y = 7 \)

(ii) What Bayes says: If \( y^0 = 6 \)

\[ y(\theta) = \sqrt{\theta_1^2 + \theta_2^2} \]

(Segments of circles)
(i) What probability says: If $y = 7$ then $r^2 \sim \chi^2$ with $2$ df and $N_C = \frac{\pi^2}{4}$

For data $y^0 = 6$ ...

Prob left of data $= H_2(6^2; \chi^2)$

$\Rightarrow$ $p$-value $= 14.1\%$

(ii) What Bayes does: If $y^0 = 6$...
(i) What probability says: If $y = 7$ then $\eta^2 \sim NC \chi^2$ with 2 df and $NC = 7^2$ for data $y^0 = 6$ ....

Prob left of data $= H_2(6^2; 7^2) = p$-value $= 14.1\%$

(ii) What Bayes says: If $y^0 = 6$ then $\theta^2 \sim NC \chi^2$ with 2 df and $NC = 6^2$

"Prob' larger than $|\theta| = 7$ "posterior survival value"

$= 1 - H_2(7^2; 6^2) = s$-value $= 17.8\%$

Not hard to see why they are different: Curvature!

-In nature of Bayes!
And... Assess $|\theta| \geq 7$ with other data points

<table>
<thead>
<tr>
<th>Data $y_i$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$-value</td>
<td>.10%</td>
<td>1.9%</td>
<td>14.1%</td>
<td>47.1%</td>
</tr>
<tr>
<td>$s$-value</td>
<td>.18%</td>
<td>2.8%</td>
<td>17.8%</td>
<td>52.9%</td>
</tr>
</tbody>
</table>

With positive curvature $s(y) > p(y)$ ... always

$\Rightarrow$ Uniformly wrong with curvature! ... Bayes is uncalibrated

"Two ways of thinking of same problem?"
"Hidden" in DSZ 1973
"Paradigm is broken!"
Curved models!

Ex: $y \sim N(\theta, \sigma^2(\theta)) \quad \sigma^2(\theta) = 1 + \gamma \theta^2 / 2n$

Linear $\gamma = 0$

Curved $\gamma > 0$

$\beta$-confidence (lower bound)

$\hat{\theta}_\beta = y - z_\beta$

$\hat{\theta}_\beta = y - z_\beta \left[1 + \gamma \frac{(y - z_\beta)^2}{4n}\right]

A "prior" cannot give this $\beta$-confidence quantile!

"... cannot convert likelihood to confidence!"

Bayes can't handle curvature!

DSZ didn't broadcast the real message

No need to use opinion priors to analyze unless statistics seems too hard!
Does it matter: Bayes can't handle curvature?

- A number is called a "probability"
- and it doesn't have the performance property
- Is it... "Wall St" deception?

Two weather models: Different measurement processes
Flat priors for parameter Simulations
Contradictory results

The Economist "Gambling on tomorrow" Aug 18, 2007:

"... the way you pick the individual values to plug into the model can cause trouble."
"... the Bayesian bolt-on does not come easily to scientists"
"... garbage in, garbage out."
"The difficulty comes when you do not know what garbage looks like"
Should Statisticians ignore this?
...in Phil Trans. Roy. Soc ?
...in The Economist ?

Elsewhere would be called Fraud

Bayes posterior: *Great* for exploring with likelihood !
  - frequentists *should* have used their own likelihood !
  and it *may* give *approximate* confidence
  - *It* does not give more than confidence
  - To claim otherwise is fraud

The Tool Box needs a lot of calibration !
Some references

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