Likelihood Overview

Context: Model \( f(y; \theta) \): Regular Asymptotic

Data \( y^0 \)

Explore: Least squares or other \( \Rightarrow \frac{\text{variable} - E}{\text{SD}} \) or Bootstrap

Confirm: \( p\)-values (\( s\)-values) \( 19/20 \) Accuracy

Primaliue: \( L(\theta) = C(f(y^0; \theta)) \) Need/want more?

Derivative: \( p\)-value (where data is re possible true \( \theta \))

\( s\)-value (where true is re examined \( \theta \))

Sometimes ... same curve flipped
Likelihood: Prob at $y^*$ under $\Theta$; logs

$$p(y^*)$$

$\hat{\Theta}$

$\log = -l_{\Theta}(\hat{\Theta})$

$\frac{n^2}{2} = \hat{\Theta} - l(\Theta)$

$n = \pm \sqrt{2(\hat{\Theta} - l)}$

$q = \frac{1}{2}(\Theta - \hat{\Theta})$

$l_{\Theta}(\hat{\Theta})$? Score

Compute for any $\Theta$

Consider as fn of $\Theta$... for theory

Data $\leftrightarrow$

Distn $\leftrightarrow$

Pvots?
Likelihood: Vector $\Theta$ Prob at $y^o$ under $\Theta$; logs

$$l(\Theta; y^o)$$

$\hat{\Theta}$

$\Sigma\Theta$

$r^2/2$ Scalars

$r \ ? \ \mathcal{N}$ ??

$$q = (\hat{\psi} \cdot \hat{\psi}) (\hat{\psi} - \hat{\psi})$$

$\theta(\theta) \ ?$

With data: Compute ... above; more?

Variable $y$: Distns for $\hat{\Theta}$, ...

Behaviour under sampling from "$\Theta$" at that $\Theta$? At other $\Theta$ values
Have $L(\theta)$ or $p(\theta)$: What to do?

a) $\int_{\theta}^{\infty} L(\theta) \pi(\theta) \, d\theta = \Delta(\theta)$

$\pi(\theta) =$ ?

Does it matter? 1st 2nd

$\hat{\psi}(\theta) > \psi^*$ ?

$\pi(\theta)$ - descriptive?

"What if?" Mathematical Interpret

b) $\int_{-\infty}^{y^*} f(y; \theta) \, dy = p(\theta)$

"How to decide where $y^* \text{ is re } \theta$?"

How to calculate? How to approximate?

Integrate - On sample space

- On parameter space

Sanctions

Merits
Case $f(y - \theta)$  

B\(|\)  

\(a)\ \pi(\theta) = c\)  

\(s(\theta) = \int_{\theta}^{\infty} f(y - \theta) \, d\theta\)  

\(p(\theta) = \int_{y}^{\infty} f(y - \theta) \, dy\)  

\| Equal

Two ways of looking at "it":

a) Choice of \(\pi(\theta)\) ? - Descriptive  
- Mathematical  
- Subjective

b) "Where is your \(y\) re \(\theta(\theta)\)?"

Does anything say: Priors frequentist calculations  

... should be combined?  
... can be combined!  
... or left separate?
Approximations: For distributions... Since CLT; since 1963
for models... Barnardoff-Nielsen?

Ex's
\[ \frac{\bar{y} - \mu}{\sigma_\bar{y}/\sqrt{n}} \] of \( N(0,1) \) / Student(\( n-1 \)) \[ O(n^{-1/2}) \] approx CLT
\[ \sqrt{n}(\hat{\theta} - \theta) \] of \( N(0,1) \) Fisher
\[ r(\theta; y) \] B-N
\[ \int_{-\infty}^{\infty} L(\theta; y) \pi(\theta) d\theta \] \[ O(n^{-1/2}) \] \( \pi(\theta) \) – subjective

Use of \( L(\theta) \) in non-Bayes ways

Are there better approximations?
1. \( f(y) \)

2. \( \log f(y) \approx O(n) \)

3. \( z = y - \hat{y} \frac{1}{Cn^{-\frac{1}{2}}} \)

4. Graph of \( \log f(z) \) vs. \( z \)

5. \( f(z) = e^{-z^2} \{ 1 + a_3 \frac{z^3}{6n^{\frac{3}{2}}} + a_4 \frac{z^4}{24n^2} + \frac{1}{2} \left( \frac{a_3^2 z^2}{\sqrt{6n^3}} \right) + O(n^{-\frac{3}{2}}) \} \)

- Work in moderate deviations
  - Conditions: Bounding \( f(z) \) outside moderate deviations
  - Use approximations models
  - Forget sufficiency etc.
Approximations (pain var: ) Who cares? "Data accretion"
Unique max "Diffuse"

Laplace: Asymptotics: \( f(y) = \pi(\theta_1 =) \Rightarrow \log[\ldots] \sim O(n) \) staged dep

\[ y \sim \frac{n^{-1}}{n^{-3/2}} \]
\[ y \sim \frac{n^{-1}}{n^{-3/2}} \]

\[ \int e^{-j y^2/2} d\theta = \frac{(2\pi)^{p/2}}{1j^{1/2}} \]

\[ \int_0^\infty f(y) dy = f(\hat{y}) \cdot \frac{(2\pi)^{p/2}}{1j^{1/2}} \left(1 + \frac{c}{n}\right) = f(\hat{y}) e^{-c/n} \]

Normal fit

Powerful (…far beyond appearances!)
Eliminate nuisance parameters in "f" analysis
Give marginal probs in Bayes calculations
Mathematical parameter … doesn't matter!

In Bayes… mostly

- Bédard & F&W
- Stat Sc 2008
- BKA 2008
- #210 AFW CSS 2005
- #209 CFMDRY 1998
- Taylor
- F & Rousseau
- JSP 1998
- Web page
Approximations (palm var:)

1. Laplace: \( f(y) \approx \) asy \( \pi(\theta = 0) \) asymptotic \( \log f(y) \sim O(n) \)
   - \( y \) staged dep
   - Unique max
   - "Diffuse"

   \[ \int e^{-j y^{3/2}} \, d\theta = \frac{(2\pi)^{p/2}}{1/j^{1/2}} \]

   Normal fit \( \hat{y} \)

   \[
   \int_{-\infty}^{\infty} f(y) \, dy = f(\hat{y}) \cdot \frac{(2\pi)^{p/2}}{1/j^{1/2}} \left(1 + \frac{c}{n}\right) = f(\hat{y}) \cdot \frac{c/n}{1/j^{1/2}}
   \]

   Powerful (far beyond appearances!)

- Eliminate nuisance parameters in "f" analysis
- Give marginal probs in Bayes calculations
- Mathematical parameter -- doesn't matter!

\( O(n^{-2}) \)

Bedard & F&W
Stat Sc 2008
F & Rousseau
Bka 2008
2. Saddlepoint (pdim var; pdim para) \[ f(y; \theta) \text{ Exponential; asymptotic} \]

\[ f = e^{\phi} \sum_{\alpha} e^{-\phi \alpha} \int h(s) \, ds \]

\[ f(s; \phi) \, ds = e^{k/n} \cdot e^{-\frac{k^2}{2}} \cdot \frac{|\int \phi \, ds|^{1/2}}{(2\pi)^{p/2}} \]

1. Only need at single \[ \phi \]

\[ a) - \frac{n}{2} = t(\phi; s) - t(\hat{\phi}; s) \]

This exactly adjusts for other \[ \phi \] values

b) \[ \phi = \phi \quad n^2 = 0 \quad \text{at center} \]

2. Change of variable

\[ \frac{d\phi}{ds} = \int \phi \, ds \]

3. Parameterization in variance

\[ 2nd = \int \phi \, ds \]

There is a local parameterization

4. From Laplace

Scalar \[ \int_{\phi} \, ds \]

\[ \phi \text{ dip } \int_{\phi} \, ds \]
\[ f(y; \theta) \, dy = e^{k/n} \frac{e^{-n^2/2}}{(2\pi)^{p/2}} \int_{|\theta|}^{1/2} d\theta \]

\[ \phi(\theta) = \frac{1}{\theta} \frac{\partial}{\partial \theta} l(\theta; y) \]

Then just Laplace/SP
Parrondo-Nielsen's $p^*$

$$f(y; \theta) = e^{\frac{k/n}{2} - \frac{y^2}{2(2\pi)^{p/2}}}$$

**Proof only at $\hat{\theta} = \hat{\theta}$, $y = \hat{y}$**

(2) Use explicit parameter $\varphi(\theta) = \frac{1}{y} \ell(\theta; y)$

Then just Laplace/SP

**Laplace: SP Version**

$$\varphi(\theta; y) = \frac{\partial}{\partial y} \ell(\theta; y)$$

Same dimension

as above

$\hat{\theta} = \hat{\theta}$

$\varphi(\theta; y) = \varphi(\theta; \hat{\theta})$

BN 1986

BN Magic formula

1) 1986 paper.

pdf for mle $\hat{\theta}$

Regularity

$R^n \rightarrow R^n \rightarrow R$

C: Assumed
4) Tangent Exponential and SP

- Laplace
- SP
- $p^*$

Integrate an arbitrary d.f.
Approx version of Exponential
SP approx more generally

Thence structure $\Rightarrow$ H.O.L.

1) $n \to p$
2) UMPS $O(n^{-\frac{1}{2}})$
Above are density/model approximations
But we want dist fn approximations!
a) - Do for scalar case
b) - Nuisance parameters eliminated by Laplace/\hat{p}^*

\[ \int e^{\phi(y)} \, dy \]

Either: Exponential (Asy); General (Asy)

\[ p(\theta) = \int_{-\infty}^{\infty} e^{\phi(y)} \, dy \]

Recall

\[ = \int_{-\infty}^{\infty} e^{\phi(y)} \, dy = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \, dy = \sqrt{2\pi} \]

\[ = \phi(n) + \varphi(n) \left( \frac{1}{n} - \frac{1}{q} \right) \text{LR} \]

\[ = \Phi(n - n^{-1} \log \frac{r}{q}) \text{BN} \]

\[ q = \text{SHLE} \text{in } \varphi \]

\[ r = \text{SLR} \]

As earlier

\[ q = \text{shape parameter} \]

\[ \text{Only at } n, q \text{ can pass \text{ para approx \ in data}} \]
Jeffreys prior: \( \pi(\theta) = \sqrt{\int I(\theta) \ d\theta} \)

- Bad properties with many models.

Regression:
\[
\sigma^{-1} \frac{d\beta}{d\sigma} \quad \sigma^{-1} \frac{d\beta}{d\sigma}
\]
Flat for \( \beta \)
Flat for \( \log \beta \)

Recent: \( \beta \) is linear \( \log \beta \) linear

Jeffreys \( \rightarrow \) mod Jeff \( \rightarrow \) Linear Components

Dawid Stone Zidek
\[ y_{1i} \sim N(\mu_1, \sigma_1^2) \]

\[ y_{2i} \sim N(\mu_2, \sigma_2^2) \]

\[ S = \mu_1 - \mu_2 \]

\[ z = \frac{y_1 - y_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \]

**Find a prior**

Problems.

**Fisher**

**Bhreens - Fisher**

Jeffreys --- mod Jeffreys

Ghosh - Kim \( \leftrightarrow \) (check: Has 2nd [\( \square \)] prior! Basis

SLR \[ \text{Train CLT + Slutsky} \]

3rd \[ p^* \text{ Integrate BN/LR} \]

**Open Issues:** Why do things go wrong with B-N? Welch - Peer approximation h = very good
For $\mu_1 = 2.0$, $\mu_2 = 0.0$, $\sigma^2 = 1.0$ without loss of generality and for various $n_1 \geq n_2$, $\sigma^2$, we record the proportion of the 10,000 cases where the true $\delta$ is less than the lower limit and less than the upper limit of the 90% central interval.

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