Digression  Distribution / "Population" on the plane

\[ y \]
\[ x \]

Joint \((x, y)\) \((\mu_x, \mu_y)\)

Marg \(x\) \(\mu_x\)

Cond \(y \mid x\) \(\mu_y(x)\) \(\sigma_y(x)\)

1) Regression of \(y\) on \(x\) = \(E(y \mid x) = \mu_2(x) = \int y \cdot f(y \mid x) \, dy = \Sigma y \cdot p(y \mid x)\)

\[ \text{fixed } x \]

2) Var about regression = \(\sigma_y^2(x) = \text{Var}(y \mid x) = \int (y - \mu_2(x))^2 \cdot f(y \mid x) \, dy.\)

Properties (1) \(E(y) = E\left[\mu_y(x)^2\right] = \left[E E\right] = E\)

\[ \text{Proof use } (y - c)^2 = (y - \mu)^2 + (\mu - c)^2 \quad SS \quad \text{Check} \]
Unbiasedness

Rao-Blackwell

Improving estimates; if α(y) is suff for β(θ); s(y) is suff

Suppose t(γ) is unb for β(θ); s(y) is suff

\[ E[t(γ); θ] = β(θ) \]

\[ = E\left\{ E[t(γ) | s] \right\}; θ^\star \neq θ \]

\[ = E\{ r(s); θ \} = β(θ) \]

Rao Blackwell Thm: t \(\rightarrow\) β(θ); s(y) SS; then r(s) = E[t(y) | s] is unb for β(θ)

and has smaller variance and strictly smaller unless t(y) is fn of S

Proof

\[ Var\{ t(γ); θ \} = E\{ \sigma^2_t(s) \} + V[r(s); θ] \]

Property 2: Say? (Original var > var of new r(s))

Actually greater unless \( E\{ \sigma^2_t(s) \} = 0 \) \(\Rightarrow\) No cond var

\[ r(s) = t(y) \]

Example: "Toy"
Ex: \( x_1, \ldots, x_n \) Bernoulli \( p \). Want to est \( p \) ? Given \( S = \sum y_i \).

Having some estimate:

\[
E(t) = p \quad \text{where} \quad t(x) = x_1 \quad E(x_1) = p = 1 \cdot p + 1 \cdot q = p
\]

Have unbiased est. of \( p \).

\[
E(t | S) = \frac{1}{n} \sum x_i = \frac{S}{n}
\]

Use R-B

\[
r(S) = E\{t | S\} = E\{x_1 | \sum x_i = S\}
\]

\[
= E\{x_1 | \sum x_i = S\}
\]

\[
= \frac{1}{n} \cdot \frac{y}{n} + 0 \cdot \frac{n-y}{n}
\]

\[
= \frac{y}{n} = \hat{p} = \text{old usual est.}
\]

Know \( \hat{p} \) is UMV (certain stuff).

Starts with \( x_1 \); got \( \hat{p} \); know \( n \) best.

\[
\frac{p \cdot q^1 \cdot q^{n-1}}{\binom{n}{1} \cdot p^n \cdot q^{n-1}} = \frac{1}{\binom{n}{1}}
\]

\[
x_1 \mid S
\]

\[
\text{Distr: } x_1 \mid S \sim \binom{n}{1} \cdot p \cdot q^{n-1}
\]

\[
\frac{p \cdot q^1 \cdot q^{n-1}}{\binom{n}{1} \cdot p^n \cdot q^{n-1}} = \frac{1}{\binom{n}{1}}
\]

\[
x_1 \mid S
\]