More on Unbiasedness \leftarrow Estimation \leftarrow Inference

Location LS \[ a(\theta) \]
Locn Error GM \[ X\beta + \epsilon \]
Full Model CR \[ b = (X'X)^{-1}X'y \]

Bartlett: 1. \[ E(\hat{\theta}; \theta) = 0 \]
2. \[ V(\hat{\theta}; \theta) = -E\{\hat{\theta} \hat{\theta}' \theta \theta' \} \]

Information Inequality... "dated" but important!

Lemma

Suppose \( t(y) \) umb for \( \beta(\theta) \)...

(\( \beta \)-Type) 3. \[ \frac{d}{d\theta} E\{t(y); \theta\} = \text{cov}\{t(y), \beta(\theta; y)\} = \beta'(\theta) \]

Proof:
\[ \int t(y) f(y; \theta) dy = \beta(\theta) \quad \forall \theta \]

\[ \frac{d}{d\theta} E(t; \theta) = \int t(y) \beta(\theta; y)f(y; \theta)dy \]

Regularity \[ E\{t(y) \beta(\theta; y); \theta\} = \text{cov}\{t(y), \beta\} \]

\[ E(\hat{\theta}) \]

Barlett #3 with \( E(\hat{t}) = \beta(\theta) \)

\[ E(x; y) \]

\[ E(x; y) \] if one of \( = 0 \)
$f(x) = ye^{-y}$ on $(0, \infty)$

$l_0(\theta) = \theta - y = \text{fn of } y$

**Fisher Information Inequality**

Have $t(y)$ unb. for a parameter $\theta$

Then $\text{cov} \{t(y), l_0(\theta; y) : \theta \}$ $\geq 1$

$$\frac{\partial^2}{\partial \theta^2} \theta = 1$$

Unbiased

$$\sqrt{\text{Unbiased}}$$

$$\text{Info Ineq } \text{Var}[t(y); \theta] \geq \frac{1}{i(\theta)}$$

$\text{t unb fn of } \theta$

and $t(y)$ is affine in $l_0$ at lower bound.

Reciprocal Info or Precision is a lower bound for variance.
Ex 1: \( y_1, \ldots, y_n \sim N(\theta, \sigma^2) \)

\[
E(\theta; y) = \frac{\bar{y} - \theta}{\sigma^2/n} \quad \text{current}
\]

\[
\text{old}(\theta; y) = \frac{n}{\sigma^2} \left( \bar{y} - \theta \right)
\]

\[
\text{current} = \frac{\bar{y} - \theta}{\sigma^2/n} \quad \text{traditional}
\]

Mathematical answer? Happens for \( \theta = \theta_0 \)

\( D(t(y)) \) is \( \text{loc. umb}(\theta_0) \) for \( \theta \) if

1. \( E[t(y); \theta_0] = \theta_0 \)
2. \( \frac{d}{d\theta} E[t(y); \theta] \bigg|_{\theta_0} = 1 \)

If \( t(y) \xrightarrow{\text{unb}} \theta \), then \( t(y) \) is \( \text{loc}(\theta_0) \text{umb} \)

Fisher Inequality: \( \text{var}(t(y)) = \frac{1}{n/\sigma^2} \quad \text{with equality if } t(y) \text{ is affine in } \theta \)

\( \bar{y} \) is \( \text{UMVU for } \theta \)

\[\text{affine fn of } \theta ??\]

\( \mathbf{J} \)

Info Ineq. (Fisher Ineq)

If \( t(y) \) is \( \text{loc. umb}(\theta_0) \) for \( \theta \) then \( \text{var}(t(y); \theta_0) \geq i(\theta_0) \)

with equality iff \( t(y) \) is affine in \( \theta \)

(Tool for umb Est)
Construct a locally unbiased with minimum var (looking for good global).

f(y; Θ); want best unbiased of Θ = scalar

Suppose that Θ near Θ₀ and see if it helps.

First try t(y) = α + c θ₀(θ₀; y) --- affine (best) at θ₀. Suppose it is loc univ? --- Is it? Can it be loc unbiased?

(i) \( E(t; θ₀) = θ₀ = a + c E\{b(θ₀; y)\} \implies a = θ₀ \)

\[ t(y) = \theta₀ + \frac{b(θ₀; y)}{i(θ₀)} \]

(ii) \( \frac{d}{dθ} E(t; θ) \big|_{θ₀} = 1 = COV\{a + c b(θ₀; y); b(θ₀; y); θ₀\} \)

\[ = E\{a + c b(θ₀) b\} \]

\[ = 0 + c \text{ Var} b = c i(θ₀) \]

\[ \implies t(y) = θ₀ + \frac{b(θ₀; y)}{i(θ₀)} \text{ is MMV for loc univ } (θ₀) \text{ of } Θ \]

Newton Raphson (close)
Example: \( y_1, \ldots, y_n \) Poisson (\( \Theta \))

\[
f(y; \Theta) = e^{-\Theta} \frac{\Theta^y}{y!}
\]

\[
e(\Theta; y) = \sum y_i \log \Theta - n \Theta + a
\]

\[
e_0(\Theta; y) = \sum y_i - n
\]

\[
e_0 = -\frac{\sum y_i}{\Theta^2}
\]

\[
I(\Theta) = \frac{n \Theta}{\Theta^2} = \frac{n}{\Theta}
\]

\[
\text{Want UMV unib for } \Theta
\]

Try mean \( \hat{\Theta}_0 \) : MVloc(lmbd Z)

\[
t(y) = \hat{\Theta}_0 + \frac{\sum y_i - \hat{\Theta}_0}{n/\Theta_0}
\]

\[
\bar{y} = \frac{\sum y_i}{n}
\]

\[\bar{y} \text{ is loc unib wrt } \Theta_0 \text{ & has min var at } \Theta_0\]

\[\text{But true for all } \Theta_0 \text{ !}\]

\[\text{So } \bar{y} \text{ is UMV unib for } \Theta_0\]

Can do more

When does it work?

Diff Ears

\[
f(y; \Theta) = e^{-\Theta} \frac{\Theta^y}{y!}
\]

\[
\psi(\Theta) t(y) + \psi(\Theta)
\]

Exp LR model

\[\text{& } t(y) \xrightarrow{unib} \frac{\psi(\Theta)}{\psi'(\Theta)}\]