Unbiasedness §9.3 and §5.3

Given a model $\beta(\theta)$, partial/full

Interest in $\beta(\theta)$

$Dt(y_j)$ is unbiased for $\beta(\theta) \leftrightarrow E[t(y_j; \theta) = \beta(\theta)]$

Example $y_1, \ldots, y_n$ i.i.d. $N(\mu, \sigma^2)$ nonparametric

$E(y_i) = \mu$ unbiased for $\mu$
$E(\sigma^2_y) = \sigma^2$ unbiased for $\sigma^2$  

But $E(\lambda_y) \neq \sigma$ not good

Example $y_1, \ldots, y_n$ i.i.d. $N(\mu, \sigma^2)$

$E(y_i) = \mu$ unbiased
$E(\sigma^2_y) = \sigma^2$ unbiased

$E(\tilde{y}_i) = \frac{(\bar{y}^2)}{(\sigma^2_y)} < \text{unbiased for } \mu$

Inherent $t(y) \text{ unbiased } \beta(\theta)$ when $\theta \in \Omega$ 

then $\frac{t_1}{\text{var}(t_1; \theta)} \text{ and } \frac{t_2}{\text{var}(t_2; \theta)} \geq \Omega$

$t_1 = 3.17 \text{ var } = (0.11)^2 \text{ Ind/}

\frac{1}{t_2 = 3.11 \text{ var } = (0.07)^2 \text{ uncon.}}

\text{ weight by precision}

\frac{1}{0.0121} \frac{1}{3.17} + \frac{1}{0.0049} \frac{1}{3.11} = 3.127

\text{ How to compare estimates?}

\text{ Have } t_1, \text{ have } t_2

Is one better than other?

$\frac{1}{\text{var}(t; \theta)}$ reciprocal

Reciprocal

$\beta(\theta) \rightarrow \frac{1}{t_1}$

$\beta(\theta) \rightarrow \frac{1}{t_2}$

Better? Small variance
\[ t_1, t_2 \text{ unable for } \beta(\theta) \]

DEFICIENCY of \( t_1 \) re \( t_2 \) = \( \frac{\text{Prec. of } t_1}{\text{Prec. of } t_2} = \frac{1}{\sqrt{n}t_1} \frac{1}{\sqrt{n}t_2} \]  

Compare precision  \( \times 2 \)

Ex: \( y_1, \ldots, y_n \sim N(\mu, \sigma^2) \)  

Interest in \( \mu \)

\[ \bar{y} = \text{mean} \quad E(\bar{y}) = \mu \quad V(\bar{y}) = \frac{\sigma^2}{n} \]

\[ \tilde{y} = \text{median} \quad E(\tilde{y}) = \mu \quad V(\tilde{y}) = \frac{\pi \sigma^2}{2n} \]

Eff(\( \bar{y} \) re \( \tilde{y} \)) = \( \frac{z_n/n \sigma^2}{n \sigma^2} = \frac{2}{\pi} = 64\% \)

Meaning of this:

\[ N \mu, \sigma^2 \text{ sampling} \]

Case A

\( \bar{y} \) with \( n = 64 \)

\[ E(\bar{y}) = \mu \]

\[ V(\bar{y}) = \frac{\sigma^2}{64} \]

Case B

\( \tilde{y} \) with \( n = 100 \)

\[ E(\tilde{y}) = \mu \]

\[ V(\tilde{y}) = \frac{\pi \sigma^2}{2.100} = \frac{\sigma^2}{64} \]

Say \( \tilde{y} \) was only 64% of info comp with \( \bar{y} \)

Work?  Van widely  \( \alpha \frac{\sigma}{\bar{y}} \)
Theorem: 1) A linear comb of \( t_1, t_2 \) that is unb. for \( \beta(\theta) \) has form

\[
t = a t_1 + (1-a) t_2 \quad (\text{weights} = 1)
\]

2) Among such the estimate with smallest variance (at \( \theta_0 \)) has

\[
a = \frac{1}{\text{var}(t_1) - \text{cov}(t_1, t_2)}
\]

\[
1-a = \frac{1}{\text{var}(t_2) - \text{cov}(t_1, t_2)}
\]

[weight by excess of var over cov]

3) If estimates are uncorrelated then

\[
a = \frac{1}{\text{var}(t_1)}
\]

\[
1-a = \frac{1}{\text{var}(t_2)}
\]

and

\[
\frac{1}{\text{var}(t; \theta_0)} = \frac{1}{\text{var}(t_1)} + \frac{1}{\text{var}(t_2)}
\]
Proof

1) \( E(\alpha t_1 + b t_2) = \alpha \beta(\theta) + b \beta(\theta) = \beta(\theta) \)
\[ \Rightarrow \alpha + b = 1 \]

2) Min - Var of \( \alpha t_1 + (1-\alpha) t_2 \)
\[ V(t) = \alpha^2 V_1 + (1-\alpha) V_2 + 2\alpha(1-\alpha)C \]

Want \( \alpha \) to get minimum... Quad
\[ \frac{dV(t)}{d\alpha} = 0 \]
End of story!

Lemma \( \begin{bmatrix} A t_1 + B t_2 \end{bmatrix} \xrightarrow{\sim} x \)
Lin comb \( t_1, t_2 \)

If \( t \) is unbiased, then \( A + B = I \)

\[ V(t) = \text{Var} \left\{ (A, B)(t_1) \right\} = (A B)(V_{11} V_{12})(A' B') \quad \text{Same formula} \]

Proof: Gauss Markov Thm