fiducial inference

Fiducial inference introduced the pivotal inversion that is central to modern confidence theory. Initially this provided confidence bounds but later was generalized to give confidence distributions on the parameter space. For this it came in direct conflict with the then prominent Bayesian approach called inverse probability. Confidence distributions are now however widespread in modern likelihood theory. Recent results from this theory indicate that the developed fiducial confidence approach is giving a consistent statement of where the parameter is with respect to the data, and indeed is consistent with recent Bayesian approaches that allow data dependent priors.

In a seminal paper, R. A. Fisher (1930) introduced the notion of fiducial inference as an alternative to what was then called inverse probability. The key step in fiducial inference is pivotal inversion, which is now standard in all of confidence theory. Fisher's example involved four pairs of observations with a concern for the correlation coefficient ρ between observations in a pair. He had available the distribution function $F(r;\rho)$ for the sample correlation coefficient r, which depends only on the population correlation ρ; and he had an observed correlation value $r^0 = .99$. He did numerical calculations with the distribution function $F(r;\rho)$, which he had himself previously derived. And he then reported (.765, 1) as a 95 per cent interval for ρ. This is fully in accord with current confidence interval theory. In present notation we would write

$$P(r < .99;\rho) = .95 = P(\hat{\rho}_{\mathrm{L}} < \rho;\rho) = P\{\rho \,\mathrm{in}\, (\hat{\rho}_{\mathrm{L}},1);\rho\},$$

where the solution of $F(r;\rho) = .95$ for ρ to obtain the parameter lower bound $\hat{\rho}_{\mathrm{L}} = \hat{\rho}_{\mathrm{L}}(r)$ is standard confidence or pivotal inversion applied to the pivot $u = F(r;\rho)$, which of course has a Uniform (0,1) distribution.

But Fisher (for example, 1930; 1933; 1935; 1956) went further and presented a distribution, called a fiducial distribution, for the parameter ρ, which as a density can be used for calculations such as

$$\int_{.765}^{1} f_{\mathrm{fid}}(\rho;r^0)\,d\rho = .95,$$

and where for the example the density has the form

$$f_{\text{fid}}(\rho; r) = -(\partial/\partial\rho)F(r; \rho);$$

this density agrees with what in recent likelihood theory would be called a confidence distribution.

But Fisher went still further and spoke of fiducial probability rather than just statements for an interval such as confidence level that we would commonly use. This attribution of probability that a parameter lies in the interval (.765, 1) attracted attack from both the inverse probability community at the time and from the more conventional community that would now be called the frequentist, and includes those having philosophical persuasions. As a consequence, many have viewed fiducial probability as wrong, and strong stigmata have been attached to it. This is rather extraordinary, given that the papers by Fisher are seminal for all of confidence theory and differ only in small deviations of presentation and development.

The key aspects of fiducial that evoked criticism are (*a*) that different pivots can lead to different distributions and thus different intervals, (*b*) that marginalization of a parameter distribution to a component parameter can give a distribution that depends on data in a way different from the obvious that would come from that data, and (*c*) that constraints on the parameter can give a distribution without total probability being equal to 1.

The alternative culture when Fisher (1930) introduced fiducial inference was inverse probability (Bayes, 1763). For this, the probability at a data point y^0, given as $f(y^0; \theta)$ and now called likelihood (Fisher, 1922) and written $L(\theta; y^0)$, is adjusted by a weight function $w(\theta)$ to give the composite

$$w(\theta)L(\theta; y^0)$$

which is then treated as an unnormed density for the parameter. The weight function $w(\theta)$ is chosen based on properties of the model and called by various names, with default prior being the most unassuming. The present rather large community using this approach is a subgroup of the Bayesian community and the approach has come to be called default Bayesian inference rather than inverse probability analysis; it can also be viewed as a routine frequentist use of the frequentist likelihood function coupled with an ad hoc weight function.

This commonly called default Bayesian approach offers great freedom for the development of statistical techniques: take an observed likelihood $L(\theta; y^0)$ based on Fisher’s (1922) proposal; attach a convenient weight function $w(\theta)$ to it; and use the composite for inference for θ. With available high-powered computers and Markov Chain Monte Carlo this leads to a wealth of possible analyses, in contrast to rather limited results from earlier frequentist approaches.

But this leads to perhaps the most influential criticism of the fiducial method (Lindley, 1958): (*d*) that a fiducial distribution is typically not an inverse probability or default Bayesian posterior.

Curiously, one finds that the default Bayesian approach is subject to precisely the same criticisms (*a*), (*b*), (*c*) that have been attached to the fiducial approach (for example, concerning (*b*), see Dawid, Stone and Zidek, 1973; see also Fraser, 1961; 1995). So the fact (*d*) that a fiducial analysis is not in general a default Bayesian analysis seems a rather hollow criticism by Lindley (1958). And of course default Bayes typically does not lead to intervals that have the confidence property. Moreover, a recently dominant interest within the current Bayesian community (Fraser and Reid, 2002) is to have methods that do reproduce in repeated sampling as do confidence intervals. Perhaps the default Bayesian community is rushing in where the frequentist community neglected its own likelihood function.

But perhaps Fisher and his fiducial approach should be given credit for the fundamental contribution of the pivotal inversion, and of giving rise to the universal confidence procedures. The change of name from fiducial to confidence and then the derogation of fiducial seem a rather heavy historical penalty to Fisher and his profound and seminal developments in statistics. Perhaps 'fiducial' did move too quickly, certainly for the times, and did neglect to develop some fine details. But the results are profound; and the default Bayesian community is finding that it cannot ignore in substance the fiducial criticisms (*a*), (*b*), (*c*); and can't avoid the repeated sampling reproducibility that is the foundation of confidence theory (*d*).

But then, how does fiducial inference work in more general contexts, particularly in the light of recent likelihood theory? For each independent coordinate, say, y_i, a pivot $z_i = h_i(y_i;\theta)$ is needed that describes with full deference to continuity how the coordinate y_i measures or provides information on the parameter θ; this pivot needs to be of the same dimension as the variable y_i and of course as implied by its name has a fixed distribution free of θ. If a coordinate is scalar, the pivot is necessarily equivalent to the distribution function $F_i(y_i;\ \theta)$ for that coordinate; if it is vector then the choice of pivot represents an explicit statement of how that coordinate variable affects the parameter and is taken as a given for the inference process. Likelihood theory then shows that the full pivot can be re-expressed to third-order accuracy in the moderate deviations region by an equivalent pivot in which the parameter θ of, say, dimension p appears in only p coordinates of the new pivot. The conditional distribution of these p coordinates given the remaining pivot coordinates (which are of course directly observable) gives effectively a new pivot with of course the same dimension as the parameter. This allows for the standard confidence pivotal inversion to produce confidence regions.

If inference focuses on a particular parameter component $\psi(\theta)$ of interest with dimension d, then the recent likelihood theory shows that the interest parameter can be isolated to third order in a d dimensional component of an equivalent pivot, and the marginal model for that pivot is otherwise free of the full parameter and provides third-order confidence regions for the interest parameter. For some background see Fraser and Reid (2001), Fraser, Reid and Wu (2001), and Fraser (2004).

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See also

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Index terms

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confidence theory

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frequentist school

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