Abstract

The estimation of signal frequency count in the presence of background noise has had much recent discussion in the physics literature, and Mandelkern [1] brings the core issues to the statistical community, in turn leading to extensive discussion by statisticians. The primary focus in [1] and in the discussion rests on confidence interval procedures. We discuss various anomalies and misleading features in this use of confidence theory, and argue that the usage is essentially decision theoretic and is being applied in a context that invites an inferential approach. We then extract what we view as the inference elements, the fundamental information available from the model and the data. This is illustrated using some simple data and some recent data from the physics literature.

1 INTRODUCTION

Mandelkern [1] brings to the statistical community a seemingly simple statistical problem that arose in high energy physics; see for example, [2], [3]. The statistical problem may appear elementary but the original problem has
substantial scientific presence: as Pekka Sinervo, a coauthor of Abe et al. [2], [3] expresses, “High energy physicists have struggled with Bayesian and frequentist perspectives, with delays of several years in certain experimental programmes hanging in the balance”.

The core statistical problem can be expressed simply. A count variable \( y \) is Poisson \((b + \mu)\), where \( b > 0 \) is known and \( \mu \geq 0 \). The goal is to extract the evidence concerning the parameter \( \mu \), particularly whether or not the parameter is greater than zero. In the physical setting the count \( y \) is viewed as the sum of \( y_1 \) counting events from background radiation, and \( y_2 \) counting events from a possible signal. In [2] and [3], the signal is the presence of a possible top quark and the data come from the collider detector at Fermilab. The background radiation count \( y_1 \) is modelled as Poisson\((b)\) and the count from the possible signal as Poisson\((\mu)\). In context there are additional aspects to the problem: for example the background mean count \( b \) is estimated, the data are obtained as subsets of more complex counts, and so on. Here however we address the simplified statistical problem on its own merits, as is largely the case in Mandelkern [1]. We write \( y \sim \text{Poisson}(b + \mu) \) and let \( \theta = b + \mu \) be the Poisson mean, so the restriction is \( \theta \geq b \).

Much statistical literature, and most of the physics proposals cited by Mandelkern [1], emphasize the construction of confidence bands for \( \theta \) at a prescribed level of confidence.

For the Poisson, a simple 95% confidence interval for \( \theta \) is given by \((\sqrt{y} \pm 0.98)\), based on \( \sqrt{y} \) being approximately distributed as \( N(\sqrt{\theta}, 0.5) \). One difficulty that this approximation ignores is the discreteness of the data. A more serious difficulty concerns the lower bound \( b \) for the parameter space; for example if \( y \) is small then the interval can be partly or completely outside the permissible range \([b, \infty)\), making nonsense of the assertion of 95% confidence. Various proposals put forward seek to modify the confidence approach to overcome these difficulties.

The discussants of Mandelkern [1] also focus on the confidence interval approach. An exception is Gleser, who suggests the use of the “likelihood function as a measure of evidence about the parameters of the model used to describe the data”; and Mandelkern in his rejoinder concurs: “it may be most appropriate to, at least in ambiguous cases, give up the notion of characterizing experimental uncertainty with a confidence interval... and to present the likelihood function for this purpose.”

It is our view that the emphasis on confidence bounds for fixed levels has led to procedures that are not satisfactory from many points of view.
In particular a confidence interval in a technical sense is derived from decision theory: we “accept” parameter values within the confidence interval and “reject” parameter values outside the interval. By contrast plotting the likelihood function is an inferential approach.

In Section 2 we record some discussion of the unified approach and its variants, and also record various anomalies associated with their use.

In Section 3 we present an inferential approach where inference elements are recorded, these being the observed likelihood function and the observed p-value function. We propose that these present the full essentials of inference concerning the parameter, and in turn allow appropriate judgments to be made concerning the parameter. An example is given using data from [3].

2 The unified approach and variants

Confidence intervals based on the theory of optimal testing can often lead to rather anomalous behavior. An optimality criterion typically involves averaging over the sample space, and in many situations there are what Fisher called ‘recognizable subsets’ of the sample space that partition the sample space. In this setting use of overall optimality can mean that intervals are constructed which effectively trade performance in a single instance for average performance in a series of instances, some of which may have recognizably different features. In particular this can give a confidence interval that is empty or a confidence interval that is the full range for the parameter: in such cases the overt confidence is clearly zero or 100% in contradiction to the prescribed or targetted confidence. For some recent discussions with examples, see Fraser [5] where fault is strongly attached to the use of optimality criteria.

The conventional Neyman intervals applied to examples with a bounded parameter space can also lead to anomalous confidence intervals. In order to avoid for the moment the issues associated with the discreteness of the Poisson distribution we let $y$ be a scalar variable with a continuous distribution stochastically increasing in a scalar parameter $\theta$. Suppose the natural ranges for these are $(-\infty, \infty)$ but that the restriction $\theta \geq b$ is prescribed. For example, this could be the Normal $(\theta, 1)$ with $\theta > b$. Let $F(y; \theta)$ and $f(y; \theta)$ be the distribution and density functions for $y$.

An optimum confidence interval derived for the unrestricted case may well lap into the inappropriate region $\theta < b$, this being the key issue in the Poisson case and mentioned for the continuous case in Mandelkern [1].
The adjustments discussed in Mandelkern [1] can be described as follows. Let \( y_L(\theta) \) and \( y_U(\theta) \) be the \( \gamma \) and \( 95\% + \gamma \) points of the \( \theta \) distribution; these form a 95\% acceptance region. Now let \( \gamma = \gamma(\theta) \) vary with \( \theta \) but of course be restricted to the interval \((0, 5\%)\). The confidence belt in the \( y \times \theta \)-space is the set union of the acceptance regions \( (y_L(\theta), y_U(\theta)) \times \{ \theta \} \); and the \( y \)-section of the two dimensional confidence belt is a 95\% confidence region and under moderate regularity will have the form \((\theta_L(y), \theta_U(y))\). The objective is to have these sets stay within the acceptable range \([b, \infty)\) by some naturally seeming choice of the adjustment function \( \gamma(\theta) \).

The likelihood ratio is used as one basis for deciding which points are to go into the acceptance interval \( (y_L(\theta), y_U(\theta)) \) and thus for determining \( \gamma(\theta) \). For inclusion in an acceptance interval the points are ordered from the largest using the ratio

\[
R = \frac{L(\tilde{\theta}; y)}{L(\theta; y)}
\]

where \( L(\theta; y) = f(y; \theta) \) and \( \tilde{\theta} = \tilde{\theta}(y) \) is a reference parameter value associated with \( y \). The Unified Approach of Feldman & Cousins [6] takes \( \hat{\theta} = \hat{\theta}(y) \) to be the maximizing \( \theta \) value as calculated in conformity with the restriction \( \theta \geq b \); in the \( \text{Normal}(\theta, 1) \) case, \( \hat{\theta} = \max(b, y) \). The New Ordering approach of Giunti [7] takes \( \hat{\theta} \) to be a Bayesian expected value for \( \theta \). Using a somewhat different starting point Mandelkern & Shultz [8] obtain likelihood from the distribution of \( \hat{\theta}(y) \), which is a marginalisation from the distribution of \( y \) itself. For the normal case this \( \hat{\theta} \) does not depend on \( y \) for \( y < b \) and not surprisingly the confidence intervals obtained by this approach are found not to depend on \( y \) for \( y < b \).

The use of these optimizing or ordering criteria can have rather insidious effects. As noted, the criteria involve shifting the acceptances to the left for low parameter values so that the 2.5\% tail probabilities on the left and the right are changed to have less on the left and more on the right; this has the effect for small data values of shifting the confidence intervals to the right, away from the excluded parameter value range. The disturbing result however is that the lower confidence bound is no longer a 2.5\% bound but something larger and perhaps undefined. And the upper confidence bound is no longer a 97.5\% bound but something larger and perhaps undefined. The confidence interval has been moved around in response to the optimizing or ordering criterion. Do the bounds of the confidence interval then have any real statistical meaning individually? And this in a context where the
location of the confidence bounds in relation to the lower limit $b$ is of fundamental and focal interest. Is this proper science to move around the interval of parameter values that get statistical sanction? We feel the use of fixed level confidence intervals particularly two sided intervals is inappropriate in the present context. In Section 3 we recommend presenting the available evidence: Saying it as it is. And then leaving the scientific issues to judgement given the evidence.

For the Poisson problem described in the introduction, Roe & Woodroofe [9] propose the use of certain conditional probabilities as the basis for the confidence belt construction following Feldman & Cousins [6]. As before let $y = y_1 + y_2$ where $y_1$ is the background Poisson $(b)$ and $y_2$ is Poisson $(\mu)$. They note that if $y = y^0$ is observed then necessarily $y_1 \leq y^0$; and accordingly they recommend the use of the conditional distribution of $y$ given $y_1 \leq y^0$, say $g(y; \mu) = f(y|y_1 \leq y^0; \mu)$ as recorded as (4) in Mandelkern [1]. However, the variable $y_1$ is not an observable variable and hence not ancillary in the usual sense. And in accord with this the proposed conditioning does not generate a partition of the sample space; this was noted in Woodroofe & Wang [10]. If $(y_1, y_2) = (0, 0)$, then $y = 0$ and it follows that the indicated conditioning is that $(y_1, y_2)$ is in the set $C = \{(0, 0), (0, 1), (0, 2), \ldots \}$ corresponding to $y_1 \leq 0$. If however we consider another point in $C$, say $(y_1, y_2) = (0, 1)$ we would have $y = 1$ and the indicated conditioning would be that $(y_1, y_2)$ is in the set $C' = \{(0, 0), (0, 1), (0, 2), \ldots , (1, 0), (1, 1), (1, 2), \ldots \}$ corresponding to $y_1 \leq 1$; this is clearly different from $C$. Thus the nominal conditional distribution does not satisfy the standard conditions for validity in describing conditional frequencies given observed information. Also not surprisingly, Mandelkern [1] notes that there is a related undercoverage which can be severe for the nominal confidence intervals constructed.

3 Likelihood and $p$-value functions

With a model function and observed data the recommendation to plot the observed likelihood has had a long presence in statistics, having appeared in Fisher [11]. Among the statistical discussants of [1], Gleser [4] was alone in recommending this approach.

As a simple example consider a sample $(y_1, \ldots, y_n)$ form the Normal $(\mu, \sigma_0^2)$. The likelihood function is $L(\mu) = c \phi(n^{1/2}(\bar{y} - \mu)/\sigma_0)$ and the $p$-value function is $\Phi(n^{1/2}(\bar{y} - \mu)/\sigma_0)$ where $\phi$ and $\Phi$ are the normal density
and distribution functions. One could reasonably plot them one above the other for ease of comparison. Because $\mu$ is a location parameter, we have the formal property that the $p$-value function is the right tail integral of the likelihood function.

We now return to the Poisson ($\theta$) with $\theta = b + \mu$ where $b$ is known and $\mu \geq 0$. The likelihood function from data $y$ is

$$L(\theta) = c\theta^y e^{-\theta}$$

where $\theta = b + \mu$. This can be plotted as a function of $\mu$ for $\mu$ in $(-b, \infty)$: for $\mu$ in $[0, \infty)$ it describes the probability at the observed data point in accord with the model, and for $\mu$ in $[-b, 0)$ it can serve as a diagnostic concerning $b$. To accommodate the discreteness we propose that the $p$-value function at the data $y$ be taken to be the interval

$$p(\theta) = \{F^-(y; \theta), F(y; \theta)\}$$

of associated numerical values, where $F(y; \theta)$ is the Poisson($\theta$) distribution function and $F^-(y; \theta)$ is the probability up to, but not including, $y$ and is given by $F(y - 1; \theta)$. Thus an observed $y$ leads in general to a continuum of numerical $p$-values for each $\theta$ being assessed. This proposal acknowledges the discreteness explicitly and yet ensures the familiar sampling property of the $p$-value function, that it have a uniform distribution on $(0, 1)$. Other aspects of the discreteness problem are addressed in Brown et al. [12] and Baker [13].

As a simple example consider $b = 2$ and data $y^0 = 3$. The likelihood and $p$-value functions are recorded in Figure 1. The $p$-value for a chosen $\mu$ is now an interval of $p$-values. The likelihood for $\mu$ is easily understood. The interpretation of the $p$-value for a given data value is exactly analogous to the percentile score on, for example, a standardized test: it expresses the percentile position of the data point relative to the parameter. For the null condition $\theta = b$ or $\mu = 0$ the $p$-value interval is $(0.677, 0.857)$.

In Abe et al. [3] after preliminary simplification from their Table 1 we have $b = 6.7$ with $y^0 = 17$. The likelihood function and $p$-value functions are plotted in Figure 2. For the null condition $\theta = b$ or $\mu = 0$ the $p$-value interval is $(0.99940, 0.99978)$ thus offering a clear statement concerning whether or not $\mu > 0$. 

6
Figure 1: The likelihood function (top) and $p$-value function (bottom) for the Poisson model, with $b = 2$ and $y^0 = 3$. For $\mu = 0$ the $p$-value interval is $(0.677, 0.857)$. 

\[
\begin{align*}
\text{likelihood} \\
0.0 & \quad 0.05 & \quad 0.10 & \quad 0.15 & \quad 0.20 \\
0.0 & \quad 0.2 & \quad 0.4 & \quad 0.6 & \quad 0.8 & \quad 1.0 \\
\end{align*}
\]

\[
\begin{align*}
mu \\
0 & \quad 5 & \quad 10 \\
\end{align*}
\]
Figure 2: The likelihood function (top) and $p$-value function (bottom) for the Poisson model, with $b = 6.7$ and $y^0 = 17$. For $\mu = 0$ the $p$-value interval is $(0.99940, 0.99978)$. 