Algebraic Extraction of the Canonical Asymptotic Model: Scalar Case

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The canonical asymptotic model provides a basis for various theoretical and computational developments. The detailed algebraic steps for the extraction of this model are developed and recorded in detail.

1. Introduction

Recent likelihood asymptotics has produced simple and highly accurate procedures for testing a scalar parameter of an $n$-dimensional continuous statistical model with $p$ parameters. The test in its structure is obtained in three stages. First, a conditioning is used to produce an appropriate $p$-dimensional variable for examining the full parameter. Second, a marginalization is used to produce a scalar pivotal quantity for assessing departure from an interest parameter value. And, third, an approximation to the distribution function of the pivotal quantity at the data point. Our focus is to extract the canonical asymptotic density such that the approximation of a distribution function corresponding to this asymptotic density can be obtained using the usual higher order asymptotic methods.

2. Centering and Rescaling

Consider a statistical model $f(y; \theta)$ with scalar $y$ and scalar $\theta$, and asymptotic properties as some background parameter $n$ becomes large. For this we assume initially that $y$ for given $\theta$ is $O_p(n^{-1/2})$ about a maximum density point and that $\ell(\theta; y) = \log f(y; \theta)$ is $O(n)$ with a unique maximum when either argument is fixed. Also let $y^0$ be an observed data point and $\hat{\theta}^0$ be the corresponding the maximum likelihood value.

Consider the Taylor series expansion of the log density $\ell(\theta; y)$ around $(\hat{\theta}^0, y^0)$,
we have

\[ \ell(\theta; y) = \sum_{i,j \geq 0} a_{ij} (\theta - \hat{\theta}^0)^i (y - y^0)^j \frac{i! j!}{i! j!}, \]  

(1)

where

\[ a_{ij} = \left. \frac{\partial^{i+j} \ell(\theta; y)}{\partial \theta^i \partial y^j} \right|_{(\hat{\theta}^0, y^0)}. \]

In particular, we keep track of expansion (1) up to the fourth degree. Then the \( a_{ij} \) matrix with \( i = 0, 1, 2, 3, 4 \) and \( j = 0, 1, 2, 3, 4 \) is

\[
\begin{bmatrix}
  a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\
  0 & a_{11} & a_{12} & a_{13} \\
  a_{20} & a_{21} & a_{22} \\
  a_{30} & a_{31} \\
  a_{40}
\end{bmatrix}
\]

where \( i \) corresponds to the parameter \( \theta \) and \( j \) to the variable \( y \). Note that \( a_{10} = 0 \) because \( \left. \frac{\partial}{\partial \theta} \ell(\theta; y) \right|_{(\hat{\theta}^0, y^0)} = 0. \)

We may have that \( \hat{\theta} \) increases with \( y \) and we do calculations in accord with this; otherwise when it decreases and \( a_{11} \) is negative, we revert to the preceding by using \( -\theta \) as the \( \hat{\theta} \) in the calculations.

For the rest of this paper, we assume \( \hat{\theta} \) increases with \( y \). We location-scale standardize the variable \( y \) and the parameter \( \theta \) and thus work in the moderate deviation range,

\[
\begin{align*}
\bar{\theta} &= (-a_{20})^{1/2} (\theta - \hat{\theta}^0) \\
\bar{y} &= (-a_{20})^{-1/2} a_{11} (y - y^0).
\end{align*}
\]

(2)

This gives observed information 1 and makes the observed gradient of the score variable also equal to 1.

From (2), we have

\[
\begin{align*}
(\theta - \hat{\theta}^0) &= (-a_{20})^{-1/2} \bar{\theta} \\
(y - y^0) &= (-a_{20})^{1/2} a_{11}^{-1} \bar{y}
\end{align*}
\]

(3)

and hence

\[ dy = (-a_{20})^{1/2} a_{11}^{-1} d\bar{y}. \]

Thus, by change of variable, the density of \( \bar{y} \) is

\[ f(\bar{y}; \bar{\theta}) d\bar{y} = f(y; \theta) dy = f(y; \theta) (-a_{20})^{1/2} a_{11}^{-1} d\bar{y}. \]

The log density can then be written as

\[ \ell(\bar{\theta}; \bar{y}) = \ell(\theta; y) + \frac{1}{2} \log(-a_{20}) - \log a_{11}. \]  

(4)
Notice that if \((\theta, y) = (\theta^0, y^0)\), then \((\tilde{\theta}, \tilde{y}) = (0, 0)\).

We first expand the left-side of (4) around \((0, 0)\) up to fourth degree and obtain

\[
\ell(\tilde{\theta}; \tilde{y}) = \tilde{a}_{00} + \tilde{a}_{01} \frac{\tilde{y}}{1!} + \tilde{a}_{02} \frac{\tilde{y}^2}{2!} + \tilde{a}_{03} \frac{\tilde{y}^3}{3!} + \tilde{a}_{04} \frac{\tilde{y}^4}{4!}
\]

\[
+ \tilde{a}_{10} \frac{\tilde{\theta}}{1!} + \tilde{a}_{11} \frac{\tilde{\theta} \tilde{y}}{1! 1!} + \tilde{a}_{12} \frac{\tilde{\theta}^2 \tilde{y}}{1! 2!} + \tilde{a}_{13} \frac{\tilde{\theta}^3 \tilde{y}}{1! 3!}
\]

\[
+ \tilde{a}_{20} \frac{\tilde{\theta}^2}{2!} + \tilde{a}_{21} \frac{\tilde{\theta}^2 \tilde{y}}{2! 1!} + \tilde{a}_{22} \frac{\tilde{\theta}^2 \tilde{y}^2}{2! 2!}
\]

\[
+ \tilde{a}_{30} \frac{\tilde{\theta}^3}{3!} + \tilde{a}_{31} \frac{\tilde{\theta}^3 \tilde{y}}{3! 1!}
\]

\[
+ \tilde{a}_{40} \frac{\tilde{\theta}^4}{4!}
\]

where \(\tilde{a}_{ij} = \frac{\partial^{i+j} \ell(\tilde{\theta}; \tilde{y})}{\partial \tilde{\theta}^i \partial \tilde{y}^j} \bigg|_{(0,0)}\)

We now expand the right-side of (4) around \((\theta^0, y^0)\) up to the fourth degree and use the relationship stated in (3). The resulting expansion is then re-expressed in terms of \(\tilde{\theta}\) and \(\tilde{y}\) and we have

\[
\ell(\theta; y) + \frac{1}{2} \log(-a_{20}) - \log(a_{11})
\]

\[
= a_{00} + \frac{1}{2} \log(-a_{20}) - \log(a_{11}) + a_{01} (-a_{20})^{1/2} a_{11}^{-1} \frac{\tilde{y}}{1!} + a_{02} (-a_{20}) a_{11}^{-2} \frac{\tilde{y}^2}{2!}
\]

\[
+ a_{03} (-a_{20})^{3/2} a_{11}^{-3} \frac{\tilde{y}^3}{3!} + a_{04} (-a_{20})^2 a_{11}^{-4} \frac{\tilde{y}^4}{4!}
\]

\[
+ \tilde{a}_{10} \frac{\tilde{\theta}}{1!} + \frac{\tilde{a}_{11} \tilde{\theta} \tilde{y}}{1! 1!} + \tilde{a}_{12} (-a_{20})^{1/2} \frac{\tilde{\theta} \tilde{y}^2}{1! 2!} + \tilde{a}_{13} (-a_{20}) a_{11}^{-3} \frac{\tilde{\theta} \tilde{y}^3}{1! 3!}
\]

\[
+ (-1) \frac{\tilde{\theta}^2}{2!} + a_{21} (-a_{20})^{-1/2} a_{11}^{-1} \frac{\tilde{\theta}^2 \tilde{y}}{2! 1!} + a_{22} a_{11}^{-2} \frac{\tilde{\theta}^2 \tilde{y}^2}{2! 2!}
\]

\[
+ a_{30} (-a_{20})^{-3/2} \frac{\tilde{\theta}^3}{3!} + a_{31} (-a_{20})^{-1} a_{11}^{-1} \frac{\tilde{\theta}^3 \tilde{y}}{3! 1!}
\]

\[
+ a_{40} (-a_{20})^{-2} \frac{\tilde{\theta}^4}{4!}
\]

Then by equating (5) and (6), we have the following equalities.

\[
\tilde{a}_{00} = a_{00} + \frac{1}{2} \log(-a_{20}) - \log a_{11}
\]

\[
\tilde{a}_{01} = (-a_{20})^{1/2} a_{11}^{-1} a_{01}
\]

\[
\tilde{a}_{02} = (-a_{20}) a_{11}^{-2} a_{02}
\]

\[
\tilde{a}_{03} = (-a_{20})^{3/2} a_{11}^{-3} a_{03}
\]

\[
\tilde{a}_{04} = (-a_{20})^2 a_{11}^{-4} a_{04}
\]

\[
\tilde{a}_{10} = 0
\]
\[
\begin{align*}
\bar{a}_{11} &= 1 \\
\bar{a}_{12} &= \left(-a_{20}\right)^{1/2}a_{11}^{-2}a_{12} \\
\bar{a}_{13} &= \left(-a_{20}\right)a_{11}^{-3}a_{13} \\
\bar{a}_{20} &= -1 \\
\bar{a}_{21} &= \left(-a_{20}\right)^{-1/2}a_{11}^{-1}a_{21} \\
\bar{a}_{22} &= a_{11}^{-2}a_{22} \\
\bar{a}_{30} &= \left(-a_{20}\right)^{-3/2}a_{30} \\
\bar{a}_{31} &= \left(-a_{20}\right)^{-1}a_{11}^{-1}a_{31} \\
\bar{a}_{40} &= \left(-a_{20}\right)^{-2}a_{40} .
\end{align*}
\]

The \( \bar{a}_{ij} \) matrix is

\[
\begin{bmatrix}
\bar{a}_{00} & \bar{a}_{01} & \bar{a}_{02} & \bar{a}_{03} & \bar{a}_{04} \\
0 & 1 & \bar{a}_{12} & \bar{a}_{13} \\
-1 & \bar{a}_{21} & \bar{a}_{22} \\
\bar{a}_{30} & \bar{a}_{31} \\
\bar{a}_{40}
\end{bmatrix}
\]

Note that \( \bar{a}_{ij} \) with \((i + j) = 3\) is \( O(n^{-1/2}) \) and with \((i + j) = 4\) is \( O(n^{-1}) \). This is important for later expansions.

For the exponential type reexpression, our aim is to find a new parameter \( \varphi \) and a new variable \( x \) such that when we expand \( \ell(\varphi; x) \) around \((\varphi^0, y^0) = (0, 0)\), we obtain

\[
\ell(\varphi; x) = \sum_{i,j \geq 0} A_{ij} \frac{\varphi^i x^j}{i! j!}
\]

where \( A_{ij} = \left. \frac{\partial^{i+j} \ell(\varphi; x)}{\partial \varphi^i \partial x^j} \right|_{(0,0)} \) and the matrix \( A_{ij} \), up to the fourth degree is:

\[
\begin{bmatrix}
A_{00} & A_{01} & A_{02} & A_{03} & A_{04} \\
0 & 1 & 0 & 0 \\
-1 & 0 & A_{22} \\
A_{30} & 0 \\
A_{40}
\end{bmatrix}
\]

Similarly, for the location type reexpression, our aim is to find a new parameter \( \beta \) and a new variable \( u \) such that when we expand \( \ell(\beta; u) \) around \((\beta^0, u^0) = (0, 0)\), we have

\[
\ell(\beta; u) = \sum_{i,j \geq 0} B_{ij} \frac{\beta^i u^j}{i! j!}
\]
where $B_{ij} = \left. \frac{\partial^{i+j} \ell(\beta; u)}{\partial \beta^i \partial u^j} \right|_{(0,0)}$ and the matrix $B_{ij}$ up to fourth degree is
\[
\begin{bmatrix}
B_{00} & B_{01} & B_{02} & B_{03} & B_{04} \\
0 & 1 & B_{30} & -B_{40} & \\
-1 & -B_{30} & B_{22} & \\
B_{30} & -B_{40} & \\
B_{40} & 
\end{bmatrix}
\]

3. Exponential type reexpression

First, we need a new parameterization,
\[
\varphi = \bar{\theta} + s_2 \frac{\bar{\theta}^2}{2!} + s_3 \frac{\bar{\theta}^3}{3!}
\]
where $s_2$ and $s_3$ and $O(n^{-1/2})$ and $O(n^{-1})$ respectively. Excluding all $O(n^{-3/2})$ or higher terms, obtain
\[
\begin{align*}
\varphi^2 &= \bar{\theta}^2 + s_2 \bar{\theta}^3 + \left(\frac{s_3}{3} + \frac{s_2^2}{4}\right) \bar{\theta}^4 \\
\varphi^3 &= \bar{\theta}^3 + \frac{3}{2} s_2 \bar{\theta}^4 \\
\varphi^4 &= \bar{\theta}^4 .
\end{align*}
\]

Now we expand $\ell(\varphi; \bar{y})$ around $(\varphi^0, \bar{y}^0) = (0, 0)$ up to the fourth degree:
\[
\ell(\varphi; \bar{y}) = M_{00} + M_{01} \frac{\varphi}{1!} + M_{02} \frac{\varphi^2}{2!} + M_{03} \frac{\varphi^3}{3!} + M_{04} \frac{\varphi^4}{4!} + M_{10} \frac{\varphi}{1!} + M_{11} \frac{\varphi \bar{y}}{1!} + M_{12} \frac{\varphi^2 \bar{y}}{2!} + M_{13} \frac{\varphi^3 \bar{y}}{3!} + M_{20} \frac{\varphi^2}{2!} + M_{21} \frac{\varphi^2 \bar{y}}{2!} + M_{22} \frac{\varphi^2 \bar{y}^2}{2!} + M_{30} \frac{\varphi^3}{3!} + M_{31} \frac{\varphi^3 \bar{y}}{3!} + M_{40} \frac{\varphi^4}{4!} 
\]
(7)
where $M_{ij} = \left. \frac{\partial^{i+j} \ell(\varphi; \bar{y})}{\partial \varphi^i \partial \bar{y}^j} \right|_{(0,0)}$

Since $\ell(\bar{\theta}; \bar{y}) = \ell(\varphi; \bar{y})$, we can compare (5) and (7). The coefficient of

- the constant term is $M_{00} = \bar{a}_{00}$
- the $\frac{\varphi}{1!}$ term is $M_{01} = \bar{a}_{01}$
- the $\frac{\varphi^2}{2!}$ term is $M_{02} = \bar{a}_{02}$
- the $\frac{\varphi^3}{3!}$ term is $M_{03} = \bar{a}_{03}$
- the $\frac{\varphi^4}{4!}$ term is $M_{04} = \bar{a}_{04}$
Also the terms that correspond to $\varphi$, $\varphi^2$, $\varphi^3$ and $\varphi^4$ are

\[
M_{10} \varphi + M_{20} \frac{\varphi^2}{2!} + M_{30} \frac{\varphi^3}{3!} + M_{40} \frac{\varphi^4}{4!}
\]

\[
= M_{10} \bar{\vartheta} + (M_{10} s_2 + M_{20}) \frac{\bar{\vartheta}^2}{2!} + (M_{10} s_3 + 3M_{20} s_2 + M_{30}) \frac{\bar{\vartheta}^3}{3!}
\]

\[
+ \left[ 12M_{20} \left( \frac{s_3}{3} + \frac{s_2^2}{4} \right) + 6M_{30} s_2 + M_{40} \right] \frac{\bar{\vartheta}^4}{4!}
\]

Hence

\[
M_{10} = \bar{a}_{10} = 0
\]

\[
M_{10}s + M_{21} = \bar{a}_{20} = -1 \Rightarrow M_{20} = -1
\]

\[
M_{10}s_3 + 3M_{20}s_2 + M_{30} = \bar{a}_{30} = \bar{a}_{30} + 3s_2
\]

\[
12M_{20} \left( \frac{s_3}{3} + \frac{s_2^2}{4} \right) + 6M_{30}s_2 + M_{40} = \bar{a}_{40}
\]

\[
\Rightarrow M_{40} = \bar{a}_{40} + 4s_3 + 3s_2^2 - 6M_{30}s_2
\]

\[
= \bar{a}_{40} + 4s_3 - 6\bar{a}_{30}s_2 - 18s_2^2
\]

The terms correspond to $\varphi \bar{\vartheta}$, $\varphi^2 \bar{\vartheta}$ and $\varphi^3 \bar{\vartheta}$ are

\[
M_{11} \varphi \bar{\vartheta} + M_{21} \frac{\varphi^2 \bar{\vartheta}}{2!} + M_{31} \frac{\varphi^3 \bar{\vartheta}}{3!}
\]

\[
= M_{11} \bar{\vartheta} \bar{\vartheta} + (M_{11}s_2 + M_{21}) \frac{\bar{\vartheta}^2 \bar{\vartheta}}{2!} + (M_{11}s_3 + 3M_{21}s_2 + M_{31}) \frac{\bar{\vartheta}^3 \bar{\vartheta}}{3!}.
\]

Hence

\[
M_{11} = \bar{a}_{11} = 1
\]

\[
M_{11}s_2 + M_{21} = \bar{a}_{21} \Rightarrow M_{21} = \bar{a}_{21} - s_2
\]

\[
M_{11}s_3 + 3M_{21}s_2 + M_{31} = \bar{a}_{31}
\]

\[
\Rightarrow M_{31} = \bar{a}_{31} - s_2 - 3M_{21}s_2
\]

\[
= \bar{a}_{31} - s_2 - 3\bar{a}_{21}s_2 + 3s_2^2
\]

The terms correspond to $\varphi \bar{\vartheta}^2$ and $\varphi^2 \bar{\vartheta}^2$ are

\[
M_{12} \frac{\varphi \bar{\vartheta}^2}{1! 2!} + M_{22} \frac{\varphi^2 \bar{\vartheta}^2}{2! 2!}
\]

\[
= \left( M_{12} \varphi + \frac{M_{22}}{2} \varphi^2 \right) \frac{\bar{\vartheta}^2}{2}
\]

\[
= \left[ M_{12} \bar{\vartheta} + M_{12} \left( \frac{\bar{\vartheta}^2}{2} \right) + M_{22} \frac{1}{2!} \bar{\vartheta}^2 \right] \frac{\bar{\vartheta}^2}{2}
\]

\[
= M_{12} \frac{\bar{\vartheta} \bar{\vartheta}^2}{1! 2!} + (M_{12}s_2 + M_{22}) \frac{\bar{\vartheta} \bar{\vartheta}^2}{2! 2!}
\]

Then

\[
M_{12} = \bar{a}_{12}
\]

\[
M_{12}s_2 + M_{22} = \bar{a}_{22} \Rightarrow M_{22} = \bar{a}_{22} - M_{12}\bar{a}_{12}
\]
And the term corresponds to $\varphi \bar{y}^3$ is

$$M_{13} \frac{\varphi \bar{y}^3}{1! \ 3!} = M_{13} \frac{\bar{\theta} \bar{y}^3}{1! \ 3!} \Rightarrow M_{13} = \bar{a}_{13}.$$ 

Since for the exponential type reexpression, $M_{21} = M_{31} = 0$, we have

$$s_2 = \bar{a}_{21}, \quad s_3 = \bar{a}_{31},$$

and thus

$$\varphi = \bar{\theta} + \bar{a}_{21} \frac{\bar{\theta}^2}{2!} + \bar{a}_{31} \frac{\bar{\theta}^3}{3!}$$

and the coefficient $M_{ij}$ in expansion (7) can be re-expressed as

$$
\begin{align*}
M_{00} &= \bar{a}_{00} \\
M_{01} &= \bar{a}_{01} \\
M_{02} &= \bar{a}_{02} \\
M_{03} &= \bar{a}_{03} \\
M_{04} &= \bar{a}_{04} \\
M_{10} &= 0 \\
M_{11} &= 1 \\
M_{12} &= \bar{a}_{12} \\
M_{13} &= \bar{a}_{13} \\
M_{20} &= -1 \\
M_{21} &= 0 \\
M_{22} &= \bar{a}_{22} - \bar{a}_{12} \bar{a}_{21} \\
M_{30} &= \bar{a}_{30} + 3 \bar{a}_{21} \\
M_{31} &= 0 \\
M_{40} &= \bar{a}_{40} + 4 \bar{a}_{31} - 6 \bar{a}_{30} \bar{a}_{21} - 15 \bar{a}_{21}^2 .
\end{align*}
$$

Now we need the new variable

$$x = \bar{y} + r_2 \frac{\bar{y}^2}{2!} + r_3 \frac{\bar{y}^3}{3!}$$

where $r_2$ and $r_3$ are $O(n^{-1/2})$ and $O(n^{-1})$ respectively. Excluding all $O(n^{-3/2})$ or higher terms, we obtain

$$
\begin{align*}
x^2 &= \bar{y}^2 + r_2 \bar{y}^3 + \left( \frac{r_3}{3} + \frac{r_2}{4} \right) \bar{y}^4 \\
x^3 &= \bar{y}^3 + \frac{3}{2} r_2 \bar{y}^4 \\
x^4 &= \bar{y}^4.
\end{align*}
$$
Also we have

\[ dx = \left( 1 + r_2 \tilde{y} + \frac{r_3}{2} \tilde{y}^2 \right) d\tilde{y}. \]

Therefore, by change of variable, we obtain

\[ f(\tilde{y}; \varphi) d\tilde{y} = f(x; \varphi) dx = f(x; \varphi) \left[ 1 + r_2 \tilde{y} + \frac{r_3}{2} \tilde{y}^2 \right] d\tilde{y} \]

and thus

\[ \ell(\varphi; \tilde{y}) = \ell(\varphi; x) + \log \left[ 1 + r_2 \tilde{y} + \frac{r_3}{2} \tilde{y}^2 \right]. \]

Note that

\[
\log \left[ 1 + \left( r_2 \tilde{y} + \frac{r_3}{2} \tilde{y}^2 \right) \right] = \left( r_2 \tilde{y} + \frac{r_3}{2} \tilde{y}^2 \right) - \frac{1}{2} \left( r_2 \tilde{y} + \frac{r_3}{2} \tilde{y}^2 \right)^2 + \ldots \\
= r_2 \tilde{y} + \left( \frac{r_3}{2} - \frac{r_3^2}{2} \right) \tilde{y}^2 + \ldots.
\]

Thus, excluding all \( O(n^{-3/2}) \) or higher order terms, we have

\[ \ell(\varphi; \tilde{y}) = \ell(\varphi; x) + r_2 \tilde{y} + \left( \frac{r_3}{2} - \frac{r_3^2}{2} \right) \tilde{y}^2. \quad (9) \]

Now we expand \( \ell(\varphi; x) \) around \((\varphi^0, x^0) = (0, 0)\)

\[
\ell(\varphi; x) = A_{00} + A_{01} \frac{x}{1!} + A_{02} \frac{x^2}{2!} + A_{03} \frac{x^3}{3!} + A_{04} \frac{x^4}{4!} \\
+ A_{10} \frac{\varphi}{1!} + A_{11} \frac{\varphi x}{1! 1!} + A_{12} \frac{\varphi^2 x^2}{1! 2!} + A_{13} \frac{\varphi^3 x^3}{1! 3!} \\
+ A_{20} \frac{\varphi^2}{2!} + A_{21} \frac{\varphi^2 x}{2! 1!} + A_{22} \frac{\varphi^2 x^2}{2! 2!} \\
+ A_{30} \frac{\varphi^3}{3!} + A_{31} \frac{\varphi^3 x}{3! 1!} \\
+ A_{40} \frac{\varphi^4}{4!}
\]

where \( A_{ij} = \frac{\partial^{i+j} \ell(\varphi; x)}{\partial \varphi^i \partial x^j} \bigg|_{(0,0)} \)

Using (9), we can compare (7) and (9). The coefficient of

the constant term is \( A_{00} = M_{00} = \tilde{a}_{00} \)

the \( \frac{\varphi}{1!} \) term is \( A_{10} = M_{10} = 0 \)

the \( \frac{\varphi^2}{2!} \) term is \( A_{20} = M_{20} = -1 \)

the \( \frac{\varphi^3}{3!} \) term is \( A_{30} = M_{30} = \tilde{a}_{30} + 3 \tilde{a}_{21} \)

the \( \frac{\varphi^4}{4!} \) term is \( A_{40} = M_{40} = \tilde{a}_{40} + 4 \tilde{a}_{31} - 6 \tilde{a}_{30} \tilde{a}_{21} - 15 \tilde{a}_{21}^2 \).
The terms that correspond to \( x, \ x^2, \ x^3 \) and \( x^4 \) are then given by

\[
A_{01} + A_{02} \frac{x^2}{2!} + A_{03} \frac{x^3}{3!} + A_{04} \frac{x^4}{4!} + r_2 \bar{y} + \left( \frac{r_3}{2} + \frac{r_2}{2} \right) \bar{y}^2
\]

\[
= r_2 \bar{y} \left( \frac{r_3}{2} + \frac{r_2}{2} \right) \bar{y}^2 + A_{01} \bar{y} + A_{01} \left( r_2 \bar{y}^2 \frac{3}{2} \right) + A_{01} \left( r_3 \bar{y}^3 \frac{3}{6} \right)
\]

\[
+ A_{02} \frac{1}{2} \bar{y}^2 + A_{02} \frac{1}{2} (r_2 \bar{y}^3) + A_{02} \frac{1}{2} \left[ \left( \frac{r_3}{3} + \frac{r_2}{4} \right) \bar{y}^4 \right] + A_{03} \frac{1}{6} (\bar{y}^3) + A_{03} \frac{1}{6} (\frac{3}{2} r_2 \bar{y}^4)
\]

\[
+ A_{04} \frac{1}{24} \bar{y}^4
\]

\[
= (r_2 + A_{01}) \bar{y} + (r_3 + r_2 + A_{01} r_2 + A_{02} \bar{y}^2 \frac{3}{2}) + (A_{01} r_3 + 3 A_{02} r_2 + A_{03} \bar{y}^3 \frac{3}{3})
\]

\[
+ \left[ 12 A_{02} \left( \frac{r_3}{3} + \frac{r_2}{4} \right) + 6 A_{03} r_2 + A_{04} \right] \bar{y}^4 \frac{4!}{4!}.
\]

Therefore

\[
r_2 + A_{01} = M_{01} \Rightarrow A_{01} = M_{01} - r_2 = \bar{a}_{01} - r_2
\]

\[
A_{01} = M_{02} - r_3 + r_2 - A_{01} r_2
\]

\[
= \bar{a}_{02} - r_3 + 2 r_2 - \bar{a}_{01} r_2
\]

\[
A_{01} r_3 + 3 A_{02} r_2 + A_{03} = M_{03}
\]

\[
\Rightarrow A_{03} = M_{03} - A_{01} r_3 - 3 A_{02} r_2
\]

\[
= \bar{a}_{03} - (\bar{a}_{01} - r_2) r_3 - 3(\bar{a}_{02} - r_3 + 2 r_2 - \bar{a}_{01} r_2) r_2
\]

\[
= \bar{a}_{03} - \bar{a}_{01} r_3 - 3 \bar{a}_{02} r_2 - 3 \bar{a}_{01} r_2^2
\]

\[
4 A_{02} r_3 + 3 A_{02} r_2^2 + 6 A_{03} r_2 + A_{04} = M_{04}
\]

\[
\Rightarrow A_{04} = M_{04} - 4 A_{02} r_3 - 3 A_{02} r_2^2 - 6 A_{03} r_2
\]

\[
= \bar{a}_{04} - 4(\bar{a}_{02} - r_3 + 2 r_2 - \bar{a}_{01} r_2) r_3
\]

\[
- 3(\bar{a}_{02} - r_3 + 2 r_2 - \bar{a}_{01} r_2) r_2
\]

\[
- 6(\bar{a}_{03} - \bar{a}_{01} r_3 - 3 \bar{a}_{02} r_2 + 3 \bar{a}_{01} r_2^2) r_2
\]

\[
= \bar{a}_{04} - 4 \bar{a}_{02} r_3 - 3 \bar{a}_{02} r_2^2 - 6 \bar{a}_{03} r_2 + 18 \bar{a}_{02} r_2^3
\]

\[
= \bar{a}_{04} - 4 \bar{a}_{02} r_3 + 15 \bar{a}_{02} r_2^2 - 6 \bar{a}_{03} r_2.
\]

The terms that correspond to \( \varphi x, \ \varphi x^2, \) and \( \varphi x^3 \) are given by

\[
A_{11} \frac{\varphi x}{1! 11} + A_{12} \frac{\varphi x^2}{2! 11} + A_{13} \frac{\varphi x^3}{3! 11}
\]

\[
= \varphi \left[ A_{11} x + A_{12} \frac{1}{2} x^2 + A_{13} \frac{1}{6} x^3 \right]
\]

\[
= \varphi \left[ A_{11} y + A_{11} (r_2 \bar{y}^2 \frac{3}{2}) + A_{11} (r_3 \bar{y}^3 \frac{3}{6})
\right.
\]

\[
+ A_{12} \frac{1}{2} (\bar{y}^2) + A_{12} \frac{1}{2} (r_2 \bar{y}^3)
\]

\[
+ A_{13} \frac{1}{6} (\bar{y}^3)
\]
\[
\begin{align*}
&= A_{11} \frac{\varphi \bar{y}}{1! 1!} + (A_{11} r_2 + A_{12}) \frac{\varphi \bar{y}^2}{1! 2!} + (A_{11} r_3 + 3A_{12} r_2 + A_{13}) \frac{\varphi \bar{y}^3}{1! 3!} . \\
\text{Therefore} \\
A_{11} &= M_{11} = 1 \\
A_{11} r_2 + A_{12} &= M_{12} \Rightarrow A_{12} = \bar{a}_{12} - r_2 \\
A_{11} + 3A_{12} + A_{13} &= M_{13} \\
\Rightarrow A_{13} &= \bar{a}_{13} - r_3 - 3(\bar{a}_{12} - r_2) = \bar{a}_{13} - r_3 - 3\bar{a}_{12} + 3r_2 . \\
\text{The terms that correspond to } \varphi^2 x \text{ and } \varphi^2 x^2 \text{ are given by} \\
A_{21} \frac{\varphi^2 x}{2! 1!} + A_{22} \frac{\varphi^2 x^2}{2! 2!} &= \frac{\varphi^2}{2!} (A_{21} x + A_{22} \frac{1}{2!} x^2) \\
&= \frac{\varphi^2}{2!} \left[ A_{21}(\bar{y}) + A_{21}(r_2 \frac{\bar{y}^2}{2!}) + A_{22} \frac{1}{2!} \bar{y}^2 \right] \\
&= A_{21} \frac{\varphi^2 \bar{y}}{2! 1!} + (A_{21} r_2 + A_{22}) \frac{\varphi^2 \bar{y}^2}{2! 2!} .
\end{align*}
\]

Therefore

\[
A_{21} = M_{21} = 0 \\
A_{21} r_2 + A_{22} = M_{22} \\
A_{22} = M_{22} = \bar{a}_{22} - \bar{a}_{12} \bar{a}_{21} .
\]

The term that corresponds \( \varphi^3 x \) is

\[
A_{31} \frac{\varphi^3 x}{3! 1!} = A_{31} \frac{\varphi^3 \bar{y}}{3! 1!} \Rightarrow A_{31} = M_{31} = 0 .
\]

Since for the exponential type re-expression, \( A_{12} = A_{13} = 0 \) and thus \( r_2 = \bar{a}_{12}, \) \( r_3 = \bar{a}_{13} \) we obtain

\[
x = \bar{y} + \bar{a}_{12} \frac{\bar{y}^2}{2!} + \bar{a}_{13} \frac{\bar{y}^3}{3!} .
\]

and expansion of \( \ell(\varphi; x) \) has coefficients \( A_{ij} \).

In summary, we have the new parameter and the new variable are

\[
\varphi = \bar{\theta} + \bar{a}_{21} \frac{\bar{\theta}^2}{2!} + \bar{a}_{31} \frac{\bar{\theta}^3}{3!} \\
x = \bar{y} + \bar{a}_{12} \frac{\bar{y}^2}{2!} + \bar{a}_{13} \frac{\bar{y}^3}{3!} 
\]

and

\[
\ell(\varphi; x) = A_{00} + A_{01} \frac{x}{1!} + A_{02} \frac{x^2}{2!} + A_{03} \frac{x^3}{3!} + A_{04} \frac{x^4}{4!}
\]
\[ +A_{10} \frac{\varphi}{1!} + A_{11} \frac{\varphi x}{1! 1!} + A_{12} \frac{\varphi x^2}{1! 2!} + A_{13} \frac{\varphi x^3}{1! 3!} + A_{20} \frac{\varphi^2}{2!} + A_{21} \frac{\varphi^2 x}{2! 1!} + A_{22} \frac{\varphi^2 x^2}{2! 2!} + A_{30} \frac{\varphi^3}{3!} + A_{31} \frac{\varphi^3 x}{3! 1!} + A_{40} \frac{\varphi^4}{4!} \]

where

\[ A_{00} = \tilde{a}_{00} \]
\[ A_{01} = \tilde{a}_{01} - \tilde{a}_{12} \]
\[ A_{02} = \tilde{a}_{02} - \tilde{a}_{01} \tilde{a}_{12} - \tilde{a}_{13} + 2\tilde{a}_{12}^2 \]
\[ A_{03} = \tilde{a}_{03} - \tilde{a}_{01} \tilde{a}_{13} - 3\tilde{a}_{02} \tilde{a}_{12} + 3\tilde{a}_{01} \tilde{a}_{12}^2 \]
\[ A_{04} = \tilde{a}_{04} - 4\tilde{a}_{02} \tilde{a}_{13} + 15\tilde{a}_{02} \tilde{a}_{12}^2 - 6\tilde{a}_{03} \tilde{a}_{12} \]
\[ A_{10} = 0 \]
\[ A_{11} = 1 \]
\[ A_{12} = 0 \]
\[ A_{13} = 0 \]
\[ A_{20} = -1 \]
\[ A_{21} = 0 \]
\[ A_{22} = \tilde{a}_{22} - \tilde{a}_{12} \tilde{a}_{21} \]
\[ A_{30} = \tilde{a}_{30} + 3\tilde{a}_{21} \]
\[ A_{31} = 0 \]
\[ A_{40} = \tilde{a}_{40} + 4\tilde{a}_{31} - 6\tilde{a}_{30} \tilde{a}_{21} - 15\tilde{a}_{21}^2 \]

As the observed likelihood is a primary ingredient, we write \( A_{30} = -\alpha_3/n^{1/2} \), \( A_{40} = -\alpha_4/n \), and \( A_{22} = c/n \) which makes explicit the asymptotic dependence on \( n \). It can then be shown that all the other coefficients are determined. We record the coefficient array now \( a = -\frac{1}{2} \log(2\pi) \). If \( c = 0 \) the model is exponential to the third order. For some background calculations see Fraser & Reid (1993), Cakmak et al (1998). A direct interpretation of the above expansion gives the tangent exponential model (Fraser & Reid, 1995) which is a generalization of the \( p^* \) formula of Barndorff-Nielsen (1983). Based on this expansion, Andrews, Fraser and Wong
(2002) obtained an approximate distribution function:

\[ F(x; \varphi) = \Phi \left[ \left( x - \varphi \right) + \frac{\alpha_{n+2}}{6n+3} \left\{ 1 - \varphi^2 - \varphi x - x^2 \right\} + \frac{\alpha_{3n+4}}{24n} \left\{ (\varphi - \varphi^3 + (3 - \varphi^2) x - \varphi x^2 - x^3 \right\} \right. \\
\left. + \frac{\alpha_{3n+6}}{72n} \left\{ (-4 \varphi + \varphi^3) + (-14 + 3 \varphi^2) x + 6 \varphi x^2 + 8 x^3 \right\} + \frac{\alpha_{4n+6}}{4n} \left\{ -2x + \varphi x^2 + x^3 \right\} \right] \]

where \( \Phi() \) is the distribution function of the standard normal distribution.

4. Location type re-expression

Similar to the exponential type re-expression, let

\[ \beta = \bar{\theta} + b_2 \bar{\theta}^2 + b_3 \bar{\theta}^3 \]

where \( b_2 \) and \( b_3 \) are \( O(n^{-1/2}) \) and \( O(n^{-1}) \) respectively. Then excluding all \( O(n^{-3/2}) \) and higher terms, we have

\[ \beta^2 = \bar{\theta}^2 + b_2 \bar{\theta}^3 + \left( \frac{b_3}{3} + \frac{b_2^2}{4} \right) \bar{\theta}^4 \]

\[ \beta^3 = \bar{\theta}^3 + \frac{3}{2} b_2 \bar{\theta}^4 \]

\[ \beta^4 = \bar{\theta}^4 \]

Now we expand \( \ell(\beta; \bar{y}) \) around \((\beta^0, \bar{y}^0) = (0, 0)\) up to the fourth degree,

\[ \ell(\beta; \bar{y}) = M_{00} + M_{01} \bar{y}^{11} + M_{02} \bar{y}^{22} + M_{03} \bar{y}^{31} + M_{04} \bar{y}^{41} + M_{05} \bar{y}^{51} + \cdots \]

\[ + M_{06} \bar{y}^{61} + M_{07} \bar{y}^{71} + M_{08} \bar{y}^{81} + M_{09} \bar{y}^{91} + M_{10} \bar{y}^{101} + \cdots \]

\[ + M_{11} \bar{y}^{11} \bar{y}^{11} + M_{12} \bar{y}^{21} \bar{y}^{11} + M_{13} \bar{y}^{31} \bar{y}^{11} + M_{14} \bar{y}^{41} \bar{y}^{11} + \cdots \]

\[ + M_{15} \bar{y}^{51} \bar{y}^{11} + M_{16} \bar{y}^{61} \bar{y}^{11} + M_{17} \bar{y}^{71} \bar{y}^{11} + M_{18} \bar{y}^{81} \bar{y}^{11} + \cdots \]

\[ + M_{19} \bar{y}^{91} \bar{y}^{11} + \cdots \]

where \( M_{ij} = \frac{\partial^{i+j} \ell(\beta; \bar{y})}{\partial \beta^i \partial \bar{y}^j} \bigg|_{(0,0)} \). Since \( \ell(\bar{\theta}; \bar{y}) = \ell(\beta; \bar{y}) \), we can compare (5) and (10). The coefficient of

- the constant term is \( M_{00} = \bar{a}_{00} \)
- the \( \bar{y}^{11} \) term is \( M_{01} = \bar{a}_{01} \)
- the \( \bar{y}^{21} \) term is \( M_{02} = \bar{a}_{02} \)
- the \( \bar{y}^{31} \) term is \( M_{03} = \bar{a}_{03} \)
- the \( \bar{y}^{41} \) term is \( M_{04} = \bar{a}_{04} \).
The terms that correspond to $\beta$, $\beta^2$, $\beta^3$ and $\beta^4$ are given by
\[
M_{10}\beta + M_{20}\frac{\beta^2}{2!} + M_{30}\frac{\beta^3}{3!} + M_{40}\frac{\beta^4}{4!}
= M_{10}\beta + M_{10}b_2\frac{1}{2}\bar{\theta}^2 + M_{10}b_2\frac{1}{6}\bar{\theta}^3
+ M_{20}\frac{1}{2}\bar{\theta}^2 + M_{20}\frac{1}{2}b_2\bar{\theta}^3 + M_{20}\frac{1}{2}\left(\frac{b_3}{3} + \frac{b_2^2}{4}\right)\bar{\theta}^4
+ M_{30}\frac{1}{6}\bar{\theta}^3 + M_{30}\frac{1}{6}b_3\bar{\theta}^4 + M_{40}\frac{1}{24}\bar{\theta}^4
= M_{10}\bar{\theta} + (M_{10}b_2 + M_{20})\frac{\bar{\theta}^2}{2!}
+ (M_{10}b_3 + 3M_{20}b_2 + M_{30})\frac{\bar{\theta}^3}{3!}
+ \left(12M_{20}\left(\frac{b_3}{3} + \frac{b_2^2}{4}\right) + 6M_{30}b_2 + M_{40}\right)\frac{\bar{\theta}^4}{4!}.
\]

Hence
\[
M_{10} = \bar{a}_{10} = 0
M_{10}b_2 + M_{20} = \bar{a}_{20} \Rightarrow M_{20} = \bar{a}_{20} = -1
M_{10}b_3 + 3M_{20}b_2 + M_{30} = \bar{a}_{30} \Rightarrow M_{30} = \bar{a}_{30} + 3b_2
+ 4M_{20}b_3 + 3M_{20}b_2^2 + 6M_{30}b_2 + M_{40} = \bar{a}_{40}
\Rightarrow -4b_3 - 3b_2^2 + 6(\bar{a}_{30} + 3b_2)b_2 + M_{40} = \bar{a}_{40}
\Rightarrow M_{40} = \bar{a}_{40} + 4b_3 - 6\bar{a}_{30}b_2 - 15b_2^2.
\]

The terms correspond to $\beta\bar{y}$, $\beta^2\bar{y}$ and $\beta^3\bar{y}$ are given by
\[
\bar{y}\left[M_{11}\beta + M_{21}\frac{\beta^2}{2!} + M_{31}\frac{\beta^3}{3!}\right]
= \bar{y}\left[+ M_{11}\frac{1}{2}b_2\bar{\theta}^2 + M_{11}\frac{1}{6}b_3\bar{\theta}^3 + M_{21}\frac{1}{2}\bar{\theta}^2 + M_{21}\frac{1}{2}b_2\bar{\theta}^3 + M_{31}\frac{1}{6}\bar{\theta}^3\right]
= \bar{y}\left[M_{11}\bar{\theta} + (M_{11}b_2 + M_{21})\frac{\bar{\theta}^2}{2!} + (M_{11}b_3 + 3M_{21}b_2 + M_{31})\frac{\bar{\theta}^3}{3!}\right].
\]

Hence
\[
M_{11} = \bar{a}_{11} = 1
M_{11}b_2 + M_{21} = \bar{a}_{21} \Rightarrow M_{21} = \bar{a}_{21} - b_2
M_{11}b_3 + 3M_{21}b_2 + M_{31} = \bar{a}_{31}
\Rightarrow b_3 + 3(\bar{a}_{21} - b_2)b_2 + M_{31} = \bar{a}_{31}
\Rightarrow M_{31} = \bar{a}_{31} - b_3 - 3\bar{a}_{21}b_2 + 3b_2^2.
\]

The terms correspond to $\beta\bar{y}^2$ and $\beta^2\bar{y}^2$ are given by
\[
\frac{\bar{y}^2}{2!}[M_{12}\beta + M_{22}\frac{\beta^2}{2}]}
\]
\[
\begin{align*}
\frac{\bar{g}^2}{2!} & \left[ M_{12} \bar{\theta} + M_{12} \frac{1}{2} b_2 \bar{\theta}^2 + M_{22} \frac{1}{2} \bar{\theta}^2 \right] \\
\frac{\bar{g}^2}{2!} & \left[ M_{12} \bar{\theta} + (M_{12} b_2 + M_{22}) \frac{\bar{\theta}^2}{2!} \right].
\end{align*}
\]

Hence

\[ M_{12} = \bar{a}_{12} \]

\[ M_{12} b_2 + M_{22} = \bar{a}_{22} \Rightarrow M_{22} = \bar{a}_{22} - \bar{a}_{12} b_2. \]

And the term corresponds to \( \beta \bar{g}^3 \) is given by

\[ \frac{\bar{g}^3}{6} [M_{13} \beta] = \frac{\bar{g}^3}{3!} M_{13} \bar{\theta} \Rightarrow M_{13} = \bar{a}_{13}. \]

Collecting all the \( M_{ij} \), we have

\[
\begin{align*}
M_{00} & = \bar{a}_{00} \\
M_{01} & = \bar{a}_{01} \\
M_{02} & = \bar{a}_{02} \\
M_{03} & = \bar{a}_{03} \\
M_{04} & = \bar{a}_{04} \\
M_{10} & = 0 \\
M_{11} & = 1 \\
M_{12} & = \bar{a}_{12} \\
M_{13} & = \bar{a}_{13} \\
M_{20} & = -1 \\
M_{21} & = \bar{a}_{21} - b_2 \\
M_{22} & = \bar{a}_{22} - \bar{a}_{12} b_2 \\
M_{30} & = \bar{a}_{30} + 3b_2 \\
M_{31} & = \bar{a}_{31} - b_3 - 3\bar{a}_{21} b_2 + 3b_2^2 \\
M_{40} & = \bar{a}_{40} + 4b_3 - 6\bar{a}_{30} b_2 - 15b_2^2.
\end{align*}
\]

Since, for the location type re-expression, \( M_{30} = -M_{21} \) and \( M_{40} = -M_{31} \), we have

\[ M_{30} = -M_{21} \Rightarrow \bar{a}_{30} + 3b_2 = -\bar{a}_{21} + b_2 \]

\[ \Rightarrow b_2 = -\frac{1}{2} (\bar{a}_{30} + \bar{a}_{21}). \]

\[ M_{40} = -M_{31} \]

\[ \Rightarrow \bar{a}_{40} + 4b_3 - 6\bar{a}_{30} b_2 - 15b_2^2 = -\bar{a}_{31} + b_3 + 3\bar{a}_{21} b_2 - 3b_2^2 \]

\[ \Rightarrow b_3 = \frac{1}{3} (3\bar{a}_{21} \bar{a}_{30} + 3\bar{a}_{21}^2 - 2\bar{a}_{31} - 2\bar{a}_{40}). \]

Let

\[ u = \bar{y} + d_2 \frac{\bar{g}^2}{2!} + d_3 \frac{\bar{g}^3}{3!}. \]
be the new variable where \( d_2 \) and \( d_3 \) are \( O(n^{-1/2}) \) and \( O(n^{-1}) \) respectively. Excluding all \( O(n^{-3/2}) \) or higher terms, we have

\[
\begin{align*}
    u^2 &= \bar{y}^2 + d_2 \bar{y}^3 + \left( \frac{d_3}{3} + \frac{d_2^2}{4} \right) \bar{y}^4 \\
    u^3 &= \bar{y}^3 + \frac{3}{2} d_2 \bar{y}^4 \\
    u^4 &= \bar{y}^4.
\end{align*}
\]

Also \( du = (1 + d_2 \bar{y} + \frac{d_3}{2} \bar{y}^2) d\bar{y} \) and as in the exponential type re-expression

\[
\ell(\beta; \bar{y}) = \ell(\beta; u) + d_2 \bar{y} + \left( \frac{d_3}{2} - \frac{d_2^2}{2} \right) \bar{y}^2.
\]  

(11)

Now expand \( \ell(\beta; u) \) around \( (\beta^0, u^0) = (0, 0) \) up to the fourth degree,

\[
\ell(\beta; u) = B_{00} + B_{01} u + B_{02} \frac{\bar{y}^2}{2!} + B_{03} \frac{u^3}{3!} + B_{04} \frac{u^4}{4!} + B_{10} \beta u + \frac{B_{11} \beta u}{1! 1!} + \frac{B_{12} \beta u^2}{1! 2!} + \frac{B_{13} \beta u^3}{1! 3!} + \frac{B_{20} \beta^2}{2!} + \frac{B_{21} \beta^2 u}{2! 1!} + \frac{B_{22} \beta^2 u^2}{2! 2!} + \frac{B_{30} \beta^3}{3!} + \frac{B_{31} \beta^3 u}{3!} + \frac{B_{40} \beta^4}{4!}.
\]

Comparing (10) and (11), we obtain the following coefficients for

- the constant term is \( B_{00} = M_{00} = \bar{a}_{00} \)
- the \( \bar{u}_1 \) term is \( B_{10} = M_{10} = 0 \)
- the \( \bar{u}_2 \) term is \( B_{20} = M_{20} = -1 \)
- the \( \bar{u}_3 \) term is \( B_{30} = M_{30} = \bar{a}_{30} + 3b_2 \)
- the \( \bar{u}_4 \) term is \( B_{40} = M_{40} = \bar{a}_{40} + 4b_3 - 6\bar{a}_{30}b_2 + 15b_2^2 \).

The terms correspond to \( u, u^2, u^3 \) and \( u^4 \) are

\[
\begin{align*}
    &B_{01} u + B_{02} \frac{u^2}{2} + B_{03} \frac{u^3}{6} + B_{04} \frac{u^4}{24} + d_2 \bar{y} + \left( \frac{d_3}{2} - \frac{d_2^2}{2} \right) \bar{y}^2 \\
    &= d_2 \bar{y} + (d_3 - d_2^2) \frac{1}{2} \bar{y}^2 + B_{01} \bar{y} + B_{02} \frac{1}{2} \bar{y}^2 + B_{03} \frac{1}{6} \bar{y}^3 + B_{04} \frac{1}{24} \bar{y}^4 \\
    &+ B_{02} \frac{1}{2} \bar{y}^2 + B_{03} \frac{1}{2} \bar{y}^3 + B_{02} \frac{1}{2} \left( \frac{d_3}{3} + \frac{d_2^2}{4} \right) \bar{y}^4 \\
    &+ B_{03} \frac{1}{6} \bar{y}^3 + B_{03} \frac{1}{6} \frac{3}{2} d_2 \bar{y}^4 + B_{04} \frac{1}{24} \bar{y}^4 \\
    &= (B_{01} + d_2) \bar{y} + (d_3 - d_2^2 + B_{01} d_2 + B_{02}) \frac{\bar{y}^2}{2!} + (B_{01} d_3 + 3 B_{02} d_2 + B_{03}) \frac{\bar{y}^3}{3!} \\
    &+ \left( 12 B_{02} \left( \frac{d_3}{3} + \frac{d_2^2}{4} \right) + 6 B_{03} d_2 + B_{04} \right) \frac{\bar{y}^4}{4!}.
\end{align*}
\]
Therefore,

\[ B_{01} + d_2 = M_{01} \Rightarrow B_{01} = \bar{a}_{01} - d_2 \]
\[ d_3 - d_2^2 + B_{01}d_2 + B_{02} = M_{02} \Rightarrow B_{02} = \bar{a}_{02} - d_3 + d_2^2 - B_{01}d_2 \]
\[ B_{01}d_3 + 3B_{02}d_2 + B_{03} = M_{03} \Rightarrow B_{03} = \bar{a}_{03} - B_{01}d_3 - 3B_{02}d_2 \]
\[ 4B_{02}d_3 + 3B_{02}d_2^2 + 6B_{03}d_2 + B_{04} = M_{04} \Rightarrow B_{04} = \bar{a}_{04} - 4B_{02}d_3 - 3B_{02}d_2^2 - 6B_{03}d_2 \]

The terms correspond to \( \beta u \), \( \beta u^2 \) and \( \beta u^3 \) are

\[
\frac{\beta}{1!} (B_{11}u + B_{12} \frac{1}{2} u^2 + B_{13} \frac{1}{6} u^3) \\
= \frac{\beta}{1!} \left[ B_{11}\bar{y} + B_{11} \frac{1}{2} d_2 \bar{y}^2 + B_{11} \frac{1}{6} d_3 \bar{y}^3 + B_{12} \frac{1}{2} \bar{y}^2 + B_{12} \frac{1}{2} d_2 \bar{y}^3 + B_{13} \frac{1}{6} \bar{y}^3 \right] \\
= \frac{\beta}{1!} \left[ B_{11}\bar{y} + (B_{11}d_2 + B_{12}) \frac{\bar{y}^2}{2!} + (B_{11}d_3 + 3B_{12}d_2 + B_{13}) \frac{\bar{y}^3}{3!} \right].
\]

Hence

\[ B_{11} = M_{11} = 1 \]
\[ B_{11}d_2 + B_{12} = M_{12} \Rightarrow B_{12} = \bar{a}_{12} - d_2B_{11}d_3 + 3B_{12}d_2 + B_{13} = M_{13} \Rightarrow B_{13} = \bar{a}_{13} - d_3 - 3B_{12}d_2 .\]

The terms correspond to \( \beta^2 u \) and \( \beta^2 u^2 \) are

\[
\frac{\beta^2}{2!} (B_{21}u + B_{22} \frac{1}{2} u^2) = \frac{\beta^2}{2!} \left[ B_{21}\bar{y} + (B_{21}d_2 + B_{22}) \frac{\bar{y}^2}{2!} \right].
\]

Therefore

\[ B_{21} = M_{21} = \bar{a}_{21} - b_2 \]
\[ B_{21}d_2 + B_{22} = M_{22} \Rightarrow B_{22} = \bar{a}_{22} - \bar{a}_{12}b_2 - B_{21}d_2 .\]

The term corresponds to \( \beta^3 u \), excluding all \( O(n^{-1/2}) \) and higher order terms, is

\[
\frac{\beta^3}{3!} B_{31}u = \frac{\beta^3}{3!} B_{31}\bar{y} \\
\Rightarrow B_{31} = M_{31} = \bar{a}_{31} - b_3 - 3\bar{a}_{21}b_2 + 3b_2^2 .\]

Collecting all the \( B_{ij} \), we have

\[
B_{00} = \bar{a}_{00} \\
B_{01} = \bar{a}_{01} - d_2 \\
B_{02} = \bar{a}_{02} - d_3 + d_2^2 - B_{01}d_2 \\
B_{03} = \bar{a}_{03} - B_{01}d_3 - 3B_{02}d_2 \\
B_{04} = \bar{a}_{04} - 4B_{02}d_3 - 3B_{02}d_2^2 - 6B_{03}d_2 .
\]
\[ B_{10} = 0 \]
\[ B_{11} = 1 \]
\[ B_{12} = \bar{a}_{12} - d_2 \]
\[ B_{13} = \bar{a}_{13} - d_3 - 3B_{12}d_2 \]
\[ B_{20} = -1 \]
\[ B_{21} = \bar{a}_{21} - b_2 \]
\[ B_{22} = \bar{a}_{22} - \bar{a}_{12}b_2 - B_{21}d_2 \]
\[ B_{30} = \bar{a}_{30} + 3b_2 \]
\[ B_{31} = \bar{a}_{31} - b_3 - 3\bar{a}_{21}b_2 + 3b_2^2 \]
\[ B_{40} = \bar{a}_{40} + 4b_3 - 6\bar{a}_{30}b_2 - 15b_2^2 \]

where

\[ b_2 = -\frac{1}{2}(\bar{a}_{30} + \bar{a}_{21}) \]
\[ b_3 = \frac{1}{6}(3\bar{a}_{21}\bar{a}_{30} + 3\bar{a}_{21}^2) - 2\bar{a}_{31} - 2\bar{a}_{40} \]

Since for the location type re-expression,

\[ B_{30} = -B_{21} = B_{12} \]
\[ B_{40} = -B_{31} = -B_{13} \]

Thus

\[ B_{12} = B_{30} \Rightarrow d_2 = \bar{a}_{12} - \bar{a}_{30} - 3b_2 \]
\[ B_{13} = -B_{40} \Rightarrow d_3 = \bar{a}_{13} - 3B_{12}d_2 + B_{40} \]

In summary, for the location type re-expression, we have the following order of calculations for the \( b_2, b_3, d_2, d_3 \) and \( B_{ij} \):

1. \( b_2 = -\frac{1}{2}(\bar{a}_{30} + \bar{a}_{21}) \)
2. \( b_3 = \frac{1}{6}(3\bar{a}_{21}\bar{a}_{30} + 3\bar{a}_{21}^2) - 2\bar{a}_{31} - 2\bar{a}_{40} \)
3. \( d_2 = \bar{a}_{12} - \bar{a}_{30} - 3b_2 \)
4. \( B_{00} = \bar{a}_{00} \)
5. \( B_{10} = 0 \)
6. \( B_{20} = -1 \)
7. \( B_{30} = \bar{a}_{30} + 3b_2 \)
8. \( B_{40} = \bar{a}_{40} + 4b_3 - 6\bar{a}_{30}b_2 - 15b_2^2 \)
9. \( B_{01} = \tilde{a}_{01} - d_2 \)
10. \( B_{11} = 1 \)
11. \( B_{21} = \tilde{a}_{21} - b_2 = -B_{30} \)
12. \( B_{31} = \tilde{a}_{40} + 4b_3 - 6\tilde{a}_{30}b_2 - 15b_2^2 = -B_{40} \)
13. \( B_{12} = B_{30} \)
14. \( d_3 = \tilde{a}_{13} - 3B_{12}d_2 + B_{40} \)
15. \( B_{02} = \tilde{a}_{02} - d_3 + d_2^2 - B_{01}d_2 \)
16. \( B_{22} = \tilde{a}_{22} - \tilde{a}_{12}b_2 - B_{21}d_2 \)
17. \( B_{03} = \tilde{a}_{03} - B_{01}d_3 - 3B_{02}d^2 \)
18. \( B_{13} = \tilde{a}_{13} - d_3 - 3B_{12}d_2 = B_{31} \)
19. \( B_{04} = \tilde{a}_{04} - 4B_{02}d_3 - 3B_{02}d^2 - 6B_{03}d_2 \)

As we are examining a log density with \( B_{30} = -\gamma_3/n^{1/2} \), \( B_{40} = -\gamma_4/n \), \( B_{22} = (-\gamma_4 + C)/n \), and \( a = -\frac{1}{2} \log(2\pi) \) we obtain the array
\[
\begin{pmatrix}
0 + (3\gamma_4 - 5\gamma_3^2 - 12C)/24n & 0 & -1 + 5C/2n & \gamma_3/n^{1/2} & (\gamma_4 - 6C)/n \\
0 & 1 & -\gamma_3/n^{1/2} & \gamma_4/n & \gamma_3/n^{-1/2} \\
-1 & \gamma_3/n^{1/2} & (-\gamma_4 + C)/n & - & - \\
-\gamma_4/n & \gamma_4/n & - & - & -
\end{pmatrix}
\]

For some background see Całmuk et al. (1998). Hence as in Andrews, Fraser and Wong (2002), we obtained an approximate distribution function:
\[
F(u; \beta) = \Phi(u - \beta) + \phi \left[ \begin{array}{c}
\frac{\tilde{a}_{01}^2}{\gamma_2} \{ -2 - (u - \beta)^2 \} \\
\frac{\tilde{a}_{01}^2}{\gamma_2} \{ 3(u - \beta) + (u - \beta)^3 \} \\
\frac{\tilde{a}_{01}^2}{\gamma_2} \{ -11(u - \beta) - (u - \beta)^3 \} \\
\frac{\tilde{C}_{01}}{\gamma_2} \{ -2u + \beta u^2 + u^3 \}
\end{array} \right]
\]

6. Examples

Example 1. Consider the scale exponential \( f(y; \theta) = \theta \exp\{-\theta y\} \) with \( \theta > 0 \), \( y > 0 \). The exact distribution function is of course available:
\[
F(y; \theta) = 1 - \exp\{-\theta y\}.
\]

Without loss of generality we take the parameter value \( \theta = \theta^0 = 1 \) and seek distribution function value for various \( y \) values around \( y^0 = 1 \) which is the point with maximum likelihood value \( \hat{\theta}(y^0) = \theta^0 = 1 \). We have
\[
\hat{\theta} = (1 - \theta) \\
\hat{y} = (y - 1).
\]
Note that we have reversed the direction for reparameterization as discussed in Section 2. This gives a new observed information equal to 1 and makes observed gradient of the score parameter also equal to 1. With

\[ x = \bar{y} \, , \, \varphi = \bar{\theta} \]

and

\[
A = \begin{bmatrix}
-1 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & - \\
-1 & 0 & 0 & - & - \\
-2 & 0 & - & - & - \\
-6 & - & - & - & -
\end{bmatrix}.
\]

In a parallel way it gives the following location reexpressions

\[ u = \bar{y} - \bar{y}^2 / 2 - 2\bar{y}^3 / 6 \, , \, \beta = \bar{\theta} + \bar{\theta}^2 / 2 + 2\bar{\theta}^3 / 6 \]

and the revised coefficient array

\[
B = \begin{bmatrix}
-1 & 0 & -1 & -1 & -1 \\
0 & 1 & 1 & 1 & - \\
-1 & -1 & -1 & - & - \\
1 & 1 & - & - & - \\
-1 & - & - & - & -
\end{bmatrix}.
\]

Table 1 records approximation values for the distribution function for \( \theta = 1 \) and selected values of \( y \) near 1.

\*Table 1: Approximate values for \( F(y; 1) \) the third order exponential expansion about \((1,1)\), the third order location expansion about \((1,1)\), and the exact value.\*

<table>
<thead>
<tr>
<th>( y )</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
<th>1.1</th>
<th>1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential</td>
<td>0.5494</td>
<td>0.5920</td>
<td>0.6306</td>
<td>0.6655</td>
<td>0.6971</td>
</tr>
<tr>
<td>Location</td>
<td>0.5497</td>
<td>0.5920</td>
<td>0.6306</td>
<td>0.6655</td>
<td>0.6973</td>
</tr>
<tr>
<td>Exact</td>
<td>0.5507</td>
<td>0.5934</td>
<td>0.6321</td>
<td>0.6672</td>
<td>0.6988</td>
</tr>
</tbody>
</table>

Both approximations give very good approximations with the exponential expansion slightly more accurate than the location expansion. Since the model is an exponential model, the third order exponential expansion could reasonably be expected to give better approximations. Note that if the \( y \) is farther away from \( y^0 = 1 \), one would expect the approximations to degrade rather quickly as they are obtained by Taylor series expansion around the data point.

\*Example 2.\* Consider the location Cauchy \( f(y; \theta) = \pi^{-1} \{1 + (y - \theta)^2\}^{-1} \) on \((-\infty, \infty)\). The exact distribution function is

\[ F(y; \theta) = .5 + \pi^{-1} \tan^{-1}(y - \theta) \]

Again we take the parameter value \( \theta^0 = 1 \) and seek distribution function values for various \( y \) values around \( y^0 = 1 \) which is the value with \( \hat{\theta}(y^0) = 1 \). We have

\[ \bar{\theta} = \sqrt{2}(\theta - 1) \]

\[ \bar{y} = \sqrt{2}(y - 1) \, . \]
and the exponential reexpressions are
\[ x = \bar{y} - \frac{3y^2}{6} \]
\[ \varphi = \bar{\theta} - \frac{3\bar{\theta}^3}{6} \]

and the revised coefficient array
\[
A = \begin{bmatrix}
-0.3466 & 0 & 2 & 0 & -9 \\
0 & 1 & 0 & 0 & - \\
-1 & 0 & 3 & - & - \\
0 & 0 & - & - & - \\
-9 & - & - & - & - \\
\end{bmatrix}
\]

In a parallel way it gives the following location reexpression
\[ u = \bar{y} \]
\[ \beta = \bar{\theta} - \frac{2\bar{\theta}^3}{6} \]

and the revised coefficient array
\[
B = \begin{bmatrix}
-0.3466 & 0 & -1 & 0 & 3 \\
0 & 1 & 0 & -3 & - \\
-1 & 0 & 3 & - & - \\
0 & -3 & - & - & - \\
3 & - & - & - & - \\
\end{bmatrix}
\]

Table 2 records approximation values for the distribution function for $\theta = 1$ and selected $y$ value near 1.

**Table 2:** Approximate values for $F(y;1)$ using the third order exponential expansion about $(1,1)$, the third order location expansion about $(1,1)$, the $y$-value being examined, and the exact value.

<table>
<thead>
<tr>
<th>$y$</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
<th>1.1</th>
<th>1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential</td>
<td>0.4297</td>
<td>0.4647</td>
<td>0.5000</td>
<td>0.5353</td>
<td>0.5703</td>
</tr>
<tr>
<td>Location</td>
<td>0.4310</td>
<td>0.4649</td>
<td>0.5000</td>
<td>0.5351</td>
<td>0.5690</td>
</tr>
<tr>
<td>Exact</td>
<td>0.4372</td>
<td>0.4683</td>
<td>0.5000</td>
<td>0.5317</td>
<td>0.5628</td>
</tr>
</tbody>
</table>

When $y$ is close to $y^0 = 1$, both approximations give very good accuracy with the location expansion slightly more accurate than the exponential expansion. As the model is a location model, we would reasonably expect the third order location expansion to give better approximations than the third order exponential expansion. Again if the $y$ is farther away from $y^0 = 1$, one would expect the approximations to degrade rather quickly as they are obtained by Taylor series expansion around the data point.
5. References


