Algebraic Extraction of the Canonical Asymptotic Model: Scalar Case

Augustine Wong
Department of Mathematics and Statistics,
York University, Toronto, Canada M3J 1P3

The canonical asymptotic model provides a basis for various theoretical and computational developments. The detailed algebraic steps for the extraction of this model are developed and recorded in detail.

1. Introduction

Consider a statistical model $f(y; \theta)$ with scalar $y$ and scalar $\theta$, and asymptotic properties as some background parameter $n$ becomes large. For this we assume initially that $y$ for given $\theta$ is $O_p(n^{-1/2})$ about a maximum density point and that $\ell(\theta; y) = \log f(y; \theta)$ is $O(n)$ with a unique maximum when either argument is fixed.

Let $y^0$ be an observed data point and $\hat{\theta}^0$ is the maximum likelihood estimate of $\theta$ corresponding to $y^0$.

2. Centering and Rescaling

Consider the relationship between $\bar{a}_{ij}$ and $a_{ij}$ as used in Section 3 in “Higher order Laplace integration and the hyper accuracy of recent likelihood methods” by D.F. Andrews, D.A.S. Fraser, and A. Wong.

Using Taylor series expansion methods, we expand $\ell(\theta; y)$ around $(\hat{\theta}^0, y^0)$ and obtain

$$
\ell(\theta; y) = \sum_{i,j \geq 0} a_{ij} \frac{(\theta - \hat{\theta}^0)^i (y - y^0)^j}{i! j!}
$$

(1)
where

\[ a_{ij} = \frac{\partial^{i+j}\ell(\theta; y)}{\partial \theta^i \partial y^j} \bigg|_{(\hat{\theta}^0, y^0)}. \]

In particular, we keep track of expansion (1) up to the fourth degree. Then the \((a_{ij})\) matrix with \(i = 0, 1, 2, 3, 4\) and \(j = 0, 1, 2, 3, 4\)

\[
\begin{bmatrix}
a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\
0 & a_{11} & a_{12} & a_{13} \\
a_{20} & a_{21} & a_{22} \\
a_{30} & a_{31} \\
a_{40}
\end{bmatrix}
\]

where \(i\) corresponds to \(\theta\) and \(j\) to the variable \(y\). Note that \(a_{10} = 0\) because \(\frac{\partial}{\partial \theta} \ell(\theta; y) \bigg|_{(\hat{\theta}^0, y^0)} = 0\).

Let

\[
\bar{\theta} = (-a_{20})^{1/2}(\theta - \hat{\theta}^0)
\]

\[
\bar{y} = (-a_{20})^{-1/2}a_{11}(y - y^0)
\]

\((\bar{\theta}, \bar{y})\) is the location-scale variables. This gives observed information 1 and makes the observed gradient of the score variable also equal to 1.

From (2), we have

\[
(\theta - \hat{\theta}^0) = (-a_{20})^{-1/2}\bar{y}
\]

\[
(y - y^0) = (-a_{20})^{1/2}a_{11}^{-1}\bar{y}
\]

and

\[
dy = (-a_{20})^{1/2}a_{11}^{-1}dy.
\]

Thus

\[
f(\bar{y}; \bar{\theta})d\bar{y} = f(y; \theta)dy = f(y; \theta)(-a_{20})^{1/2}a_{11}^{-1}d\bar{y}
\]

and we have

\[
\ell(\bar{\theta}; \bar{y}) = \ell(\theta; y) + \frac{1}{2} \log(-a_{20}) - \log a_{11}.
\]

Notice that if \((\theta, y) = (\hat{\theta}^0, y^0)\), then \((\bar{\theta}, \bar{y}) = (0, 0)\).
Now expand left-side of (4) around \((0,0)\) up to fourth degree; we obtain

\[
\ell(\bar{\theta}; \bar{y}) = a_{00} + a_{01} \frac{\bar{y}}{1!} + a_{02} \frac{\bar{y}^2}{2!} + a_{03} \frac{\bar{y}^3}{3!} + a_{04} \frac{\bar{y}^4}{4!},
\]

\[
= a_{10} \frac{\bar{\theta}}{1!} + a_{11} \frac{\bar{\theta} \bar{y}}{1! 1!} + a_{12} \frac{\bar{\theta} \bar{y}^2}{2! 1!} + a_{13} \frac{\bar{\theta} \bar{y}^3}{3! 1!}.
\]

\[
= a_{20} \frac{\bar{\theta}^2}{2!} + a_{21} \frac{\bar{\theta}^2 \bar{y}}{2! 1!} + a_{22} \frac{\bar{\theta}^2 \bar{y}^2}{2! 2!}.
\]

\[
= a_{30} \frac{\bar{\theta}^3}{3!} + a_{31} \frac{\bar{\theta}^3 \bar{y}}{3! 1!}.
\]

\[
= a_{40} \frac{\bar{\theta}^4}{4!} + \cdots
\]

where \(\bar{a}_{ij} = \frac{\partial^{i+j} \ell(\bar{\theta}; \bar{y})}{\partial \bar{\theta}^i \partial \bar{y}^j} \bigg|_{(0,0)}\).

Expand the right-side of (4) around \((\hat{\theta}^0, y^0)\) up to the fourth degree and using (3) re-express the expansion in terms of \(\bar{\theta}\) and \(\bar{y}\); we obtain

\[
\ell(\theta; y) + \frac{1}{2} \log(-a_{20}) - \log(a_{11})
\]

\[
= [a_{00} + \frac{1}{2} \log(-a_{20}) - \log(a_{11}) + a_{01} \frac{(y-y^0)}{1!} + a_{02} \frac{(y-y^0)^2}{2!} + a_{03} \frac{(y-y^0)^3}{3!} + a_{04} \frac{(y-y^0)^4}{4!}]
\]

\[
= a_{10} \frac{(\theta - \hat{\theta}^0)}{1!} + a_{11} \frac{(\theta - \hat{\theta}^0)(y-y^0)}{1! 1!} + a_{12} \frac{(\theta - \hat{\theta}^0)(y-y^0)^2}{1! 2!} + a_{13} \frac{(\theta - \hat{\theta}^0)(y-y^0)^3}{1! 3!} + a_{20} \frac{(\theta - \hat{\theta}^0)^2}{2!} + a_{21} \frac{(\theta - \hat{\theta}^0)^2(y-y^0)}{2! 1!} + a_{22} \frac{(\theta - \hat{\theta}^0)^2(y-y^0)^2}{2! 2!} + a_{30} \frac{(\theta - \hat{\theta}^0)^3}{3!} + a_{31} \frac{(\theta - \hat{\theta}^0)^3(y-y^0)}{3! 1!} + a_{40} \frac{(\theta - \hat{\theta}^0)^4}{4!} + \cdots]
\]

\[
= [a_{00} + \frac{1}{2} \log(-a_{20}) - \log(a_{11}) + a_{01} (-a_{20})^{1/2} a_{11}^{-1} \frac{\bar{y}}{1!} + a_{02} (-a_{20}) a_{11}^{-2} \frac{\bar{y}^2}{2!} + a_{03} (-a_{20})^{3/2} a_{11}^{-3} \frac{\bar{y}^3}{3!} + a_{04} (-a_{20})^2 a_{11}^{-4} \frac{\bar{y}^4}{4!} + a_{10} (-a_{20})^{-1/2} \frac{\bar{\theta}}{1!} + a_{11} (-a_{20})^{-1/2} \frac{\bar{\theta} \bar{y}}{1! 1!} + a_{12} (-a_{20})^{-1/2} \frac{\bar{\theta} \bar{y}^2}{1! 2!} + a_{13} (-a_{20})^{-1/2} \frac{\bar{\theta} \bar{y}^3}{1! 3!} + a_{20} (-a_{20})^{-1/2} \frac{\bar{\theta}^2}{2!} + a_{21} (-a_{20})^{-1/2} \frac{\bar{\theta}^2 \bar{y}}{2! 1!} + a_{22} (-a_{20})^{-1/2} \frac{\bar{\theta}^2 \bar{y}^2}{2! 2!} + a_{23} (-a_{20})^{-1/2} \frac{\bar{\theta}^2 \bar{y}^3}{2! 3!} + a_{24} (-a_{20})^{-1/2} \frac{\bar{\theta}^2 \bar{y}^4}{2! 4!} + \cdots]
\]
\[ + a_{22}(-a_{20})^{-1}(-a_{20})a_{11}^{-2} \bar{\theta}^2 \bar{y}^2 \]
\[ + a_{30}(-a_{20})^{-3/2} \bar{\theta}^3 \]
\[ + a_{31}(-a_{20})^{-3/2}(-a_{20})^{-1/2}a_{11}^{-1} \bar{\theta} \bar{y} \]
\[ + a_{40}(-a_{20})^{-2} \bar{\theta}^4 \]
\[ + \ldots \]
\[ = \left[ a_{00} + \frac{1}{2} \log(-a_{20}) - \log a_{11} \right] + a_{01}(-a_{20})^{1/2}a_{11}^{-1} \bar{\theta} \bar{y} + a_{02}(-a_{20})a_{11}^{-2} \bar{\theta}^3 \]
\[ + a_{03}(-a_{20})^{3/2}a_{11}^{-3} \bar{\theta}^3 \]
\[ + a_{04}(-a_{20})^{2}a_{11}^{-4} \bar{\theta}^4 \]
\[ + a_{05}(-a_{20})^{1} \bar{\theta} \bar{y} \]
\[ + a_{12}(-a_{20})^{-1/2}a_{11}^{-1} \bar{\theta} \bar{y} \]
\[ + a_{21}(-a_{20})^{-1/2}a_{11}^{-2} \bar{\theta} \bar{y} \]
\[ + a_{30}(-a_{20})^{-3/2} \bar{\theta} \bar{y} \]
\[ + a_{31}(-a_{20})^{-1/2}a_{11}^{-1} \bar{\theta} \bar{y} \]
\[ + a_{40}(-a_{20})^{-2} \bar{\theta} \bar{y} \]
\[ + \ldots \]  \( (6) \)

Then by equating (5) and (6), we obtain equation (3.3) of Andrews, Fraser, Wong (loc. 14):

\[ \bar{a}_{00} = a_{00} + \frac{1}{2} \log(-a_{20}) - \log a_{11} \]
\[ \bar{a}_{01} = (-a_{20})^{1/2}a_{11}^{-1}a_{01} \]
\[ \bar{a}_{02} = (-a_{20})a_{11}^{-2}a_{02} \]
\[ \bar{a}_{03} = (-a_{20})^{3/2}a_{11}^{-3}a_{03} \]
\[ \bar{a}_{04} = (-a_{20})^{2}a_{11}^{-4}a_{04} \]
\[ \bar{a}_{10} = 0 \]
\[ \bar{a}_{11} = 1 \]
\[ \bar{a}_{12} = (-a_{20})^{1/2}a_{11}^{-2}a_{12} \]
\[ \bar{a}_{13} = (-a_{20})a_{11}^{-3}a_{13} \]
\[ \bar{a}_{21} = (-a_{20})^{-1/2}a_{11}^{-1}a_{21} \]
\[ \bar{a}_{22} = a_{11}^{-2}a_{22} \]

4
\[ \bar{a}_{30} = (-a_{20})^{-3/2}a_{30} \]
\[ \bar{a}_{31} = (-a_{20})^{-1/2}a_{11}^{-1}a_{31} \]
\[ \bar{a}_{40} = (-a_{20})^{-2}a_{40} . \]

Then the \( \left( \bar{a}_{ij} \right) \) matrix is

\[
\begin{bmatrix}
\bar{a}_{00} & \bar{a}_{01} & \bar{a}_{02} & \bar{a}_{03} & \bar{a}_{04} \\
0 & 1 & \bar{a}_{12} & \bar{a}_{13} \\
-1 & \bar{a}_{21} & \bar{a}_{22} \\
\bar{a}_{30} & \bar{a}_{31} & \\
\bar{a}_{40} & \\
\end{bmatrix}
\]

Note that \( \bar{a}_{ij} \) with \( (i+j) = 3 \) is \( O(n^{-1/2}) \) and with \( (i+j) = 4 \) is \( O(n^{-1}) \). This is important for later expansions. For the exponential type reexpression, our aim is to find a new parameter \( \varphi \) and a new variable \( x \) such that when we expand \( \ell(\varphi; x) \) around \( (\varphi^0, y^0) = (0, 0) \), we have

\[
\ell(\varphi; x) = \sum_{i,j \geq 0} A_{ij} \frac{\varphi^i x^j}{i! j!}
\]

where \( A_{ij} = \frac{\partial^{i+j} \ell(\varphi; x)}{\partial \varphi^i \partial x^j} \bigg|_{(0,0)} \) and the matrix \( \left( \left( A_{ij} \right) \right) \), up to the fourth degree is:

\[
\begin{bmatrix}
A_{00} & A_{01} & A_{02} & A_{03} & A_{04} \\
0 & 1 & 0 & 0 \\
-1 & 0 & A_{22} \\
A_{30} & 0 & \\
A_{40} & \\
\end{bmatrix}
\]

Similarly, for the location type reexpression, our aim is to find a new parameter \( \beta \) and a new variable \( u \) such that when we expand \( \ell(\beta; u) \) around \( (\beta^0, u^0) = (0, 0) \), we have

\[
\ell(\beta; u) = \sum_{i,j \geq 0} B_{ij} \frac{\beta^i u^j}{i! j!}
\]

where \( B_{ij} = \frac{\partial^{i+j} \ell(\beta; u)}{\partial \beta^i \partial u^j} \bigg|_{(0,0)} \) and the matrix \( \left( \left( B_{ij} \right) \right) \) up to fourth degree is

\[
\begin{bmatrix}
B_{00} & B_{01} & B_{02} & B_{03} & B_{04} \\
0 & 1 & B_{30} & -B_{40} \\
-1 & -B_{30} & B_{22} \\
B_{30} & -B_{40} & \\
B_{40} & \\
\end{bmatrix}
\]
3. Exponential type reexpression

First, we need a new parameterization,

\[
\varphi = \bar{\varphi} + s_2 \frac{\bar{\varphi}^2}{2!} + s_3 \frac{\bar{\varphi}^3}{3!}
\]

where \( s_2 \) and \( s_3 \) and \( O(n^{-1/2}) \) and \( O(n^{-1}) \) respectively. Excluding all \( O(n^{-3/2}) \) or higher terms, we obtain

\[
\begin{align*}
\varphi^2 &= \bar{\varphi}^2 + s_2 \bar{\varphi}^3 + \left( \frac{s_3}{3} + \frac{s_2^2}{4} \right) \bar{\varphi}^4 \\
\varphi^3 &= \bar{\varphi}^3 + \frac{3}{2} s_2 \bar{\varphi}^4 \\
\varphi^4 &= \bar{\varphi}^4 .
\end{align*}
\]

Now we expand \( \ell(\varphi; \bar{y}) \) around \((\varphi^0, \bar{y}^0) = (0, 0)\),

\[
\ell(\varphi; \bar{y}) = M_{00} + M_{01} \frac{\bar{y}}{1!} + M_{02} \frac{\bar{y}^2}{2!} + M_{03} \frac{\bar{y}^3}{3!} + M_{04} \frac{\bar{y}^4}{4!} \\
+ M_{10} \frac{\varphi}{1!} + M_{11} \frac{\varphi \bar{y}}{1! 1!} + M_{12} \frac{\varphi \bar{y}^2}{1! 2!} + M_{13} \frac{\varphi \bar{y}^3}{1! 3!} \\
+ M_{20} \frac{\varphi^2}{2!} + M_{21} \frac{\varphi^2 \bar{y}}{2! 1!} \\
+ M_{30} \frac{\varphi^3}{3!} + M_{31} \frac{\varphi^3 \bar{y}}{3! 1!} \\
+ M_{40} \frac{\varphi^4}{4!} + \cdots
\]

(7)

where \( M_{ij} = \frac{\partial^{i+j} \ell(\varphi; \bar{y})}{\partial \varphi^i \partial \bar{y}^j} \bigg|_{(0,0)} \).

Since \( \ell(\bar{\varphi}; \bar{y}) = \ell(\varphi; \bar{y}) \), we can compare (5) and (7). The coefficient of

the constant term is \( M_{00} = \bar{a}_{00} \)

the \( \frac{\bar{y}}{1!} \) term is \( M_{01} = \bar{a}_{01} \)

the \( \frac{\bar{y}^2}{2!} \) term is \( M_{02} = \bar{a}_{02} \)

the \( \frac{\bar{y}^3}{3!} \) term is \( M_{03} = \bar{a}_{03} \)

the \( \frac{\bar{y}^4}{4!} \) term is \( M_{04} = \bar{a}_{04} \).
Also the terms that correspond to \( \varphi, \varphi^2, \varphi^3 \) and \( \varphi^4 \) are

\[
M_{10}\varphi + M_{20}\frac{\varphi^2}{2!} + M_{30}\frac{\varphi^3}{3!} + M_{40}\frac{\varphi^4}{4!} = M_{10} + M_{10}\left(\frac{s^2}{2}\right) + M_{10}\left(\frac{s^3}{6}\right) + M_{20}\frac{1}{2}\left(\frac{s}{2}\right) + M_{20}\frac{1}{2}\left(\frac{s^2}{2}\right) + M_{20}\frac{1}{2}\left(\frac{s}{2}\right) + M_{30}\frac{1}{6}\left(\frac{s^2}{2}\right) + M_{30}\frac{1}{6}\left(\frac{s^3}{2}\right) + M_{40}\frac{1}{24}\left(\frac{s^4}{2}\right)
\]

\[
= M_{10}\tilde{\vartheta} + (M_{10}s_2 + M_{20})\frac{\tilde{\vartheta}^2}{2!} + (M_{10}s_3 + 3M_{20}s_2 + M_{30})\frac{\tilde{\vartheta}^3}{3!} + 12M_{20}\left(\frac{s^3}{3} + \frac{s^2}{4}\right) + 6M_{30}s_2 + M_{40}\frac{\tilde{\vartheta}^4}{4!}
\]

Hence

\[M_{10} = \tilde{a}_{10} = 0\]

\[M_{10}s + M_{21} = \tilde{a}_{20} = -1 \Rightarrow M_{20} = -1\]

\[M_{10}s_3 + 3M_{20}s_2 + M_{30} = \tilde{a}_{30}\]

\[\Rightarrow M_{30} = \tilde{a}_{30} + 3s_2\]

\[12M_{20}\left(\frac{s^3}{3} + \frac{s^2}{4}\right) + 6M_{30}s_2 + M_{40} = \tilde{a}_{40}\]

\[\Rightarrow M_{40} = \tilde{a}_{40} + 4s_3 + 3s_2^2 - 6M_{30}s_2\]

\[= \tilde{a}_{40} + 4s_3 - 6\tilde{a}_{30}s_2 - 15s_2^2\]

The terms correspond to \( \varphi\tilde{y}, \varphi^2\tilde{y} \) and \( \varphi^3\tilde{y} \) are

\[
M_{11}\varphi\tilde{y} + M_{21}\frac{\varphi^2\tilde{y}}{2!} + M_{31}\frac{\varphi^3\tilde{y}}{3!} = [M_{11}\varphi + M_{21}\frac{\varphi^2}{2} + M_{31}\frac{\varphi^3}{3}]\tilde{y}
\]

\[= \tilde{y}\left[M_{11}\tilde{\vartheta} + M_{11}\left(\frac{s_2}{2}\right) + M_{11}\left(\frac{s^2}{6}\right) + M_{21}\frac{1}{2}\left(\frac{s}{2}\right) + M_{21}\frac{1}{2}\left(\frac{s^2}{2}\right) + M_{31}\frac{1}{6}\left(\frac{s^2}{2}\right)\right]
\]

\[= M_{11}\tilde{\vartheta}\tilde{y} + (M_{11}s_2 + M_{21})\frac{\tilde{\vartheta}^2\tilde{y}}{2!} + (M_{11}s_3 + 3M_{21}s_2 + M_{31})\frac{\tilde{\vartheta}^3\tilde{y}}{3!}
\]

7
Hence
\[ M_{11} = \bar{a}_{11} = 1 \]
\[ M_{11} s_2 + M_{21} = \bar{a}_{21} \Rightarrow M_{21} = \bar{a}_{21} - s_2 \]
\[ M_{11} s_3 + 3M_{21} s_2 + M_{31} = \bar{a}_{31} \]
\[ \Rightarrow M_{31} = \bar{a}_{31} - s_3 - 3M_{21} s_2 \]
\[ = \bar{a}_{31} - s_3 - 3\bar{a}_{21} s_2 + 3s_2^2 . \]

The terms correspond to \( \varphi \tilde{y}^2 \) and \( \varphi \tilde{y}^2 \) are
\[
M_{12} \frac{\varphi \tilde{y}^2}{1! 2!} + M_{22} \frac{\varphi^2 \tilde{y}^2}{2! 2!} \\
= (M_{12} \varphi + M_{22} \frac{1}{2} \varphi^2) \frac{\tilde{y}^2}{2} \\
= \frac{\tilde{y}^2}{2} \left[ M_{12} \tilde{\varphi} + M_{12} \left( s_2 \frac{\tilde{\varphi}^2}{2} \right) \right] \\
+ M_{22} \frac{1}{2} \tilde{\varphi}^2 \\
= M_{12} \frac{\tilde{\varphi} \tilde{y}^2}{1! 2!} + (M_{12} s_2 + M_{22}) \frac{\tilde{\varphi}^2 \tilde{y}^2}{2! 2!} \\
\]

Then
\[ M_{12} = \bar{a}_{12} \]
\[ M_{12} s_2 + M_{22} = \bar{a}_{22} \Rightarrow M_{22} = \bar{a}_{22} - M_{12} \bar{a}_{12} . \]

And the term corresponds to \( \varphi \tilde{y}^3 \) is
\[
M_{13} \frac{\varphi \tilde{y}^3}{1! 3!} = M_{13} \frac{\tilde{\varphi} \tilde{y}^3}{1! 3!} \\
\Rightarrow M_{13} = \bar{a}_{13} . \]

Since for the exponential type reexpression, \( M_{21} = M_{31} = 0 \)
\[ \Rightarrow s_2 = \bar{a}_{21} \]
\[ s_3 = \bar{a}_{31} \]

therefore we
\[ \varphi = \tilde{\varphi} + \bar{a}_{21} \frac{\tilde{\varphi}^2}{2!} + \bar{a}_{31} \frac{\tilde{\varphi}^3}{3!} \] (8)
and expansion (7) has coefficients. $M_{ij}$ where

$$M_{00} = \tilde{a}_{00}$$
$$M_{01} = \tilde{a}_{01}$$
$$M_{02} = \tilde{a}_{02}$$
$$M_{03} = \tilde{a}_{03}$$
$$M_{04} = \tilde{a}_{04}$$
$$M_{10} = 0$$
$$M_{11} = 1$$
$$M_{12} = \tilde{a}_{12}$$
$$M_{13} = \tilde{a}_{13}$$
$$M_{20} = -1$$
$$M_{21} = 0$$
$$M_{22} = \tilde{a}_{22} - \tilde{a}_{12}\tilde{a}_{21}$$
$$M_{30} = \tilde{a}_{30} + 3\tilde{a}_{21}$$
$$M_{31} = 0$$
$$M_{40} = \tilde{a}_{40} + 4\tilde{a}_{31} - 6\tilde{a}_{30}\tilde{a}_{21} - 15\tilde{a}_{21}^2 .$$

Now we need the new variable

$$x = \bar{y} + \sqrt{2}\tilde{y}^2 + \frac{\sqrt{3}\tilde{y}^3}{2!}$$

where $\sqrt{2}$ and $\sqrt{3}$ are $O(n^{-1/2})$ and $O(n^{-1})$ respectively. Excluding all $O(n^{-3/2})$ or higher terms, we obtain

$$x^2 = \bar{y}^2 + \sqrt{2}\tilde{y}^3 + \left(\frac{\sqrt{3}}{3} + \frac{\sqrt{2}^2}{4}\right)\tilde{y}^4$$
$$x^3 = \bar{y}^3 + \frac{3}{2}\sqrt{2}\tilde{y}^4$$
$$x^4 = \bar{y}^4 .$$
Also we have

\[ dx = \left( 1 + \sqrt{2\bar{y}} + \frac{\sqrt{3}}{2} \bar{y}^2 \right) d\bar{y}. \]

Therefore, by change of variable

\[ f(\bar{y}; \varphi)d\bar{y} = f(x; \varphi)dx \]

\[ = f(x; \varphi) \left[ 1 + \sqrt{2\bar{y}} + \frac{\sqrt{3}}{2} \bar{y}^2 \right] d\bar{y} \]

and thus

\[ \ell(\varphi; \bar{y}) = \ell(\varphi; x) + \log \left[ 1 + \sqrt{2\bar{y}} + \frac{\sqrt{3}}{2} \bar{y}^2 \right]. \]

Note that

\[ \log \left[ 1 + \left( \sqrt{2\bar{y}} + \frac{\sqrt{3}}{2} \bar{y}^2 \right) \right] = \left( \sqrt{2\bar{y}} + \frac{\sqrt{3}}{2} \bar{y}^2 \right) - \frac{1}{2} \left( \sqrt{2\bar{y}} + \frac{\sqrt{3}}{2} \bar{y}^2 \right)^2 + \cdots \]

\[ = \sqrt{2\bar{y}} + \left( \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \bar{y}^2 \right) + \cdots. \]

Thus

\[ \ell(\varphi; \bar{y}) = \ell(\varphi; x) + \sqrt{2\bar{y}} + \left( \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \bar{y}^2 \right) \bar{y}^2. \]  

(9)

Now we expand \( \ell(\varphi; x) \) around \((\varphi^0, x^0) = (0, 0)\)

\[
\ell(\varphi; x) = A_{00} + A_{01} \frac{x}{1!} + A_{02} \frac{x^2}{2!} + A_{03} \frac{x^3}{3!} + A_{04} \frac{x^4}{4!} + \]

\[ + A_{10} \frac{\varphi}{1!} + A_{11} \frac{\varphi x}{1!} + A_{12} \frac{\varphi x^2}{1! 2!} + A_{13} \frac{\varphi x^3}{1! 3!} + \]

\[ + A_{20} \frac{\varphi^2}{2!} + A_{21} \frac{\varphi^2 x}{2! 1!} + A_{22} \frac{\varphi^2 x^2}{2! 2!} + \]

\[ + A_{30} \frac{\varphi^3}{3!} + A_{31} \frac{\varphi^3 x}{3! 1!} + \]

\[ + A_{40} \frac{\varphi^4}{4!} \]

where \( A_{ij} = \frac{\partial^{i+j} \ell(\varphi; x)}{\partial \varphi^i \partial x^j} \bigg|_{(0,0)} \).
Using (9), we can compare (7) and (9). The coefficient of
the constant term is \( A_{00} = M_{00} = \bar{a}_{00} \)
the \( \frac{\varphi}{1!} \) term is \( A_{10} = M_{10} = 0 \)
the \( \frac{\varphi^2}{2!} \) term is \( A_{20} = M_{20} = -1 \)
the \( \frac{\varphi^3}{3!} \) term is \( A_{30} = M_{30} = \bar{a}_{30} + 3\bar{a}_{21} \)
the \( \frac{\varphi^4}{4!} \) term is \( A_{40} = M_{40} = \bar{a}_{40} + 4\bar{a}_{31} - 6\bar{a}_{30}\bar{a}_{21} - 15\bar{a}_{21}^2 \).
The terms that correspond to \( x \), \( x^2 \), \( x^3 \) and \( x^4 \) are given by

\[
A_{01} + A_{02}\frac{x^2}{2!} + A_{03}\frac{x^3}{3!} + A_{04}\frac{x^4}{4!} + \sqrt{2}\bar{y} + \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\right)\bar{y}^2
= \sqrt{2}\bar{y} + \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\right)\bar{y}^2
+ A_{01}\bar{y} + A_{01}\left(\sqrt{2}\bar{y}^2\right)
+ A_{01}\left(\sqrt{3}\bar{y}^3\right)
+ A_{02}\left(\frac{1}{2}\right)(\bar{y}^2)
+ A_{02}\left(\frac{1}{2}\right)(\sqrt{2}\bar{y}^3)
+ A_{02}\left(\frac{1}{2}\right)\left[\left(\frac{\sqrt{3}}{3} + \frac{\sqrt{2}}{4}\right)\bar{y}^4\right]
+ A_{03}\left(\frac{1}{6}\right)(\bar{y}^3)
+ A_{03}\left(\frac{1}{6}\right)\left(\frac{3}{2}\sqrt{2}\bar{y}^4\right)
+ A_{04}\frac{1}{24}\bar{y}^4
\]

\[
= \left(\sqrt{2} + A_{01}\right)\bar{y} + \left(\sqrt{3} + 2\sqrt{2} + A_{01}\sqrt{2} + A_{02}\right)\frac{\bar{y}^2}{2!}
+ (A_{01}\sqrt{3} + 3A_{02}\sqrt{2} + A_{03})\frac{\bar{y}^3}{3!}
+ \left[12A_{02}\left(\frac{\sqrt{3}}{3} + \frac{\sqrt{2}}{4}\right) + 6A_{03}\sqrt{2} + A_{04}\right]\frac{\bar{y}^4}{4!}
\]

Therefore

\[
\sqrt{2} + A_{01} = M_{01} \Rightarrow A_{01} = M_{01} - \sqrt{2} = \bar{a}_{01} - \sqrt{2}
\]

\[
\sqrt{3} + \sqrt{2}^2 + A_{01}\sqrt{2} + A_{02} = M_{02} \Rightarrow A_{02} = M_{02} - \sqrt{3} + \sqrt{2}^2 - A_{01}\sqrt{2}
= \bar{a}_{02} - \sqrt{3} + 2\sqrt{2}^2 - \bar{a}_{01}\sqrt{2}
\]

\[
A_{01}\sqrt{3} + 3A_{02}\sqrt{2} + A_{03} = M_{03}
\]

\[
\Rightarrow A_{03} = M_{03} - A_{01}\sqrt{3} - 3A_{02}\sqrt{2}
= \bar{a}_{03} - (\bar{a}_{01} - \sqrt{2})\sqrt{3} - 3(\bar{a}_{02} - \sqrt{3} + 2\sqrt{2}^2 - \bar{a}_{01}\sqrt{2})\sqrt{2}
= \bar{a}_{03} - 3\sqrt{3} - 3\bar{a}_{02}\sqrt{2} + 3\bar{a}_{01}\sqrt{2}^2
\]
(excluding all higher order terms)

11
\[4A_{02}\sqrt{3} + 3A_{02}\sqrt{2} + 6A_{03}\sqrt{2} + A_{04} = M_{04}\]
\[\Rightarrow A_{04} = M_{04} - 4A_{02}\sqrt{3} - 3A_{02}\sqrt{2} - 6A_{03}\sqrt{2}\]
\[= a_{04} - 4(a_{02} - \sqrt{3} + 2\sqrt{2} - a_{01}\sqrt{2})\sqrt{3}\]
\[= 3(a_{02} - \sqrt{3} + 2\sqrt{2} - a_{01}\sqrt{2})\sqrt{2}\]
\[= 6(a_{03} - a_{01}\sqrt{3} - 3a_{02}\sqrt{2} + 3a_{01}\sqrt{2})\sqrt{2}\]
\[= a_{04} - 4a_{02}\sqrt{3} - 3a_{02}\sqrt{2} - 6a_{03}\sqrt{2} + 18a_{02}\sqrt{2}\]
\[= a_{04} - 4a_{02}\sqrt{3} + 15a_{02}\sqrt{2} - 6a_{03}\sqrt{2} .\]

The terms that correspond to \(\varphi x\), \(\varphi x^2\), and \(\varphi x^3\) are given by
\[
\begin{align*}
A_{11} \frac{\varphi x}{1!} &+ A_{12} \frac{\varphi x^2}{2!} &+ A_{13} \frac{\varphi x^3}{3!} \\
= \varphi(A_{11}x) &+ A_{12} \frac{1}{2}x^2 &+ A_{13} \frac{1}{6}x^3 \\
= \varphi \left(A_{11}y\right) &+ A_{11}\left(\sqrt{2}\bar{y}^2\right) &+ A_{11}\left(\sqrt{3}\bar{g}^3\right) \\
&+ A_{12} \frac{1}{2}(\bar{y}^2) &+ A_{12} \frac{1}{2}\left(\sqrt{2}\bar{g}^3\right) \\
&+ A_{13} \frac{1}{6}(\bar{g}^3) \\
= A_{11} \frac{\varphi \bar{y}}{1!} &+ (A_{11}\sqrt{2} + A_{12}) \frac{\varphi \bar{y}^2}{2!} &+ (A_{11}\sqrt{3} + 3A_{12}\sqrt{2} + A_{13}) \frac{\varphi \bar{g}^3}{3!} .
\end{align*}
\]

Therefore
\[A_{11} = M_{11} = 1\]
\[A_{11}\sqrt{2} + A_{12} = M_{12} \Rightarrow A_{12} = a_{12} - \sqrt{2}\]
\[A_{11} + 3A_{12} + A_{13} = M_{13} \Rightarrow A_{13} = a_{13} - \sqrt{3} - 3(a_{12} - \sqrt{2})\]
\[= a_{13} - \sqrt{3} - 3a_{12} + 3\sqrt{2} .\]

The terms that correspond to \(\varphi^2 x\) and \(\varphi^2 x^2\) are given by
\[
\begin{align*}
A_{21} \frac{\varphi^2 x}{2!} &+ A_{22} \frac{\varphi^2 x^2}{2!} \\
= \frac{\varphi^2}{2} (A_{21}x &+ A_{22} \frac{1}{2}x^2) \\
= \frac{\varphi^2}{2} \left[A_{21}(\bar{y}) + A_{21}\left(\sqrt{2}\bar{g}^2\right) &+ A_{22} \frac{1}{2}\bar{g}^2\right] \\
= A_{21} \frac{\varphi^2 \bar{y}}{2!} &+ (A_{21}\sqrt{2} + A_{22}) \frac{\varphi^2 \bar{g}^2}{2!} \\
= 12.
\end{align*}
\]
Therefore

\[ A_{21} = M_{21} = 0 \]
\[ A_{21} \sqrt{2} + A_{22} = M_{22} \]
\[ A_{22} = M_{22} = \bar{a}_{22} - \bar{a}_{12} \bar{a}_{21} \, . \]

The term that corresponds \( \varphi^3 x \) is

\[ A_{31} \frac{\varphi^3 x}{3!} = A_{31} \frac{\varphi^3 y}{3!} \]
\[ \Rightarrow A_{31} = M_{31} = 0 \, . \]

Since for the exponential type re-expression, \( A_{12} = A_{13} = 0 \) and thus \( \sqrt{2} = \bar{a}_{12}, \sqrt{3} = \bar{a}_{13} \) we obtain

\[ x = \bar{y} + \bar{a}_{12} \frac{\bar{y}^2}{2!} + \bar{a}_{13} \frac{\bar{y}^3}{3!} \]

and expansion of \( \ell(\varphi; x) \) has coefficients \( A_{i j} \).

In summary, we have the new parameter and the new variable

\[ \varphi = \bar{\theta} + \bar{a}_{21} \frac{\bar{\theta}^2}{2!} + \bar{a}_{31} \frac{\bar{\theta}^3}{3!} \]
\[ x = \bar{y} + \bar{a}_{12} \frac{\bar{y}^2}{2!} + \bar{a}_{13} \frac{\bar{y}^3}{3!} \]

and

\[ \ell(\varphi; x) = A_{00} + A_{01} \frac{x}{1!} + A_{02} \frac{x^2}{2!} + A_{03} \frac{x^3}{3!} + A_{04} \frac{x^4}{4!} \]
\[ + A_{10} \frac{\varphi}{1!} + A_{11} \frac{\varphi x}{1! 1!} + A_{12} \frac{\varphi x^2}{1! 2!} + A_{13} \frac{\varphi x^3}{1! 3!} \]
\[ + A_{20} \frac{\varphi^2}{2!} + A_{21} \frac{\varphi^2 x}{2! 1!} + A_{22} \frac{\varphi^2 x^2}{2! 2!} \]
\[ + A_{30} \frac{\varphi^3}{3!} + A_{31} \frac{\varphi^3 x}{3! 1!} \]
\[ + A_{40} \frac{\varphi^4}{4!} \]
where

\[ A_{00} = \bar{a}_{00} \]
\[ A_{01} = \bar{a}_{01} - \bar{a}_{12} \]
\[ A_{02} = \bar{a}_{02} - \bar{a}_{01}\bar{a}_{12} - \bar{a}_{13} + 2\bar{a}_{12}^2 \]
\[ A_{03} = \bar{a}_{03} - \bar{a}_{01}\bar{a}_{13} - 3\bar{a}_{02}\bar{a}_{12} + 3\bar{a}_{01}\bar{a}_{12}^2 \]
\[ A_{04} = \bar{a}_{04} - 4\bar{a}_{02}\bar{a}_{13} + 15\bar{a}_{02}\bar{a}_{12}^2 - 6\bar{a}_{03}\bar{a}_{12} \]
\[ A_{10} = 0 \]
\[ A_{11} = 1 \]
\[ A_{12} = 0 \]
\[ A_{13} = 0 \]
\[ A_{20} = -1 \]
\[ A_{21} = 0 \]
\[ A_{22} = \bar{a}_{22} - \bar{a}_{12}\bar{a}_{21} \]
\[ A_{30} = \bar{a}_{30} + 3\bar{a}_{21} \]
\[ A_{31} = 0 \]
\[ A_{40} = \bar{a}_{40} + 4\bar{a}_{31} - 6\bar{a}_{30}\bar{a}_{21} - 15\bar{a}_{21}^2 \]

Then we obtain equation (3.6) of the paper.

4. Location type re-expression

Similar to the exponential type re-expression, let

\[ \beta = \bar{\theta} = b_2 \frac{\bar{\theta}^2}{2!} + b_3 \frac{\bar{\theta}^3}{3!} \]

where \( b_2 \) and \( b_3 \) are \( O(n^{-1/2}) \) and \( O(n^{-1}) \) respectively. Then excluding all \( O(n^{-3/2}) \)
and higher terms, we have

\[
\begin{align*}
\beta^2 &= \bar{\theta}^2 + b_2 \bar{\theta}^3 + \left( \frac{b_3}{3} + \frac{b_2^2}{4} \right) \bar{\theta}^4 \\
\beta^3 &= \bar{\theta}^3 + \frac{3}{2} b_2 \bar{\theta}^4 \\
\beta^4 &= \bar{\theta}^4.
\end{align*}
\]

Now we expand \( \ell(\beta; \bar{\theta}) \) around \((\beta^0, \bar{\theta}^0) = (0, 0)\),

\[
\ell(\beta; \bar{\theta}) = M_{00} + M_{01} \frac{\bar{\theta}}{1!} + M_{02} \frac{\bar{\theta}^2}{2!} + M_{03} \frac{\bar{\theta}^3}{3!} + M_{04} \frac{\bar{\theta}^4}{4!} \\
+ M_{11} \frac{\beta \bar{\theta}}{1! 1!} + M_{12} \frac{\beta \bar{\theta}^2}{1! 2!} + M_{13} \frac{\beta \bar{\theta}^3}{1! 3!} \\
+ M_{20} \frac{\beta^2}{1!} + M_{21} \frac{\beta^2 \bar{\theta}}{2! 1!} + M_{22} \frac{\beta^2 \bar{\theta}^2}{2! 2!} \\
+ M_{30} \frac{\beta^3}{3!} + M_{31} \frac{\beta^3 \bar{\theta}}{3! 1!} \\
+ M_{40} \frac{\beta^4}{4!} + \cdots
\]  

(10)

where \( M_{ij} = \frac{\partial^i + j \ell(\beta; \bar{\theta})}{\partial \beta^i \partial \bar{\theta}^j} \bigg|_{(0,0)} \). Since \( \ell(\bar{\theta}; \bar{\theta}) = \ell(\beta; \bar{\theta}) \), we can compare (5) and (10).

The coefficient of

the \( \bar{\theta} \) constant term is \( M_{00} = \bar{a}_{00} \)

the \( \frac{\bar{\theta}}{1!} \) term is \( M_{01} = \bar{a}_{01} \)

the \( \frac{\bar{\theta}^2}{2!} \) term is \( M_{02} = \bar{a}_{02} \)

the \( \frac{\bar{\theta}^3}{3!} \) term is \( M_{03} = \bar{a}_{03} \)

the \( \frac{\bar{\theta}^4}{4!} \) term is \( M_{04} = \bar{a}_{04} \).
The terms that correspond to $\beta$, $\beta^2$, $\beta^3$ and $\beta^4$ are:

$$M_{10}\beta + M_{20}\frac{\beta^2}{2!} + M_{30}\frac{\beta^3}{3!} + M_{40}\frac{\beta^4}{4!}$$

$$= M_{10}\beta + M_{10}b_2\frac{1}{2}\tilde{\beta}^2 + M_{10}b_6\frac{1}{6}\tilde{\beta}^3$$

$$+ M_{20}\frac{1}{2}\tilde{\beta}^2 + M_{20}\frac{1}{2}b_2\tilde{\beta}^3 + M_{20}\frac{1}{2}\left(\frac{b_3}{3} + \frac{b_2^2}{4}\right)\tilde{\beta}^4$$

$$+ M_{30}\frac{1}{6}\tilde{\beta}^3 + M_{30}\frac{1}{6}b_2\tilde{\beta}^4 + M_{40}\frac{1}{24}\tilde{\beta}^4$$

$$= M_{10}\tilde{\beta} + (M_{10}b_2 + M_{20})\frac{\tilde{\beta}^2}{2!}$$

$$+ (M_{10}b_3 + 3M_{20}b_2 + M_{30})\frac{\tilde{\beta}^3}{3!}$$

$$+ \left(12M_{20}\left(\frac{b_3}{3} + \frac{b_2^2}{4}\right) + 6M_{30}b_2 + M_{40}\right)\frac{\tilde{\beta}^4}{4!}.$$  

Hence

$$M_{10} = \bar{a}_{10} = 0$$

$$M_{10}b_2 + M_{20} = \bar{a}_{20} \quad \Rightarrow \quad M_{20} = \bar{a}_{20} = -1$$

$$M_{10}b_3 + 3M_{20}b_2 + M_{30} = \bar{a}_{30}$$

$$\Rightarrow \quad M_{30} = \bar{a}_{30} + 3b_2$$

$$4M_{20}b_3 + 3M_{20}b_2^2 + 6M_{30}b_2 + M_{40} = \bar{a}_{40}$$

$$- 4b_3 - 3b_2^2 + 6(\bar{a}_{30} + 3b_2)b_2 + M_{40} = \bar{a}_{40}$$

$$\Rightarrow \quad M_{40} = \bar{a}_{40} + 4b_3 - 6\bar{a}_{30}b_2 - 15b_2^2.$$  

The terms correspond to $\beta\tilde{\beta}$, $\beta^2\tilde{\beta}$ and $\beta^3\tilde{\beta}$ are

$$\tilde{\beta}\left[M_{11}\beta + M_{21}\frac{\beta^2}{2!} + M_{31}\frac{\beta^3}{3!}\right]$$

$$= \tilde{\beta}\left[ + M_{11}\frac{1}{2}b_2\tilde{\beta}^2 + M_{11}\frac{1}{6}b_2^2\tilde{\beta}^3ight.$$ 

$$+ M_{21}\frac{1}{2}\tilde{\beta}^2 + M_{21}b_2\tilde{\beta}^3$$

$$+ M_{31}\frac{1}{6}\tilde{\beta}^3\right]$$

$$= \tilde{\beta}\left[M_{11}\tilde{\beta} + (M_{11}b_2 + M_{21})\frac{\tilde{\beta}^2}{2!} + (M_{11}b_3 + 3M_{21}b_2 + M_{31})\frac{\tilde{\beta}^3}{3!}\right].$$

16
Hence
\[ M_{11} = \bar{a}_{11} = 1 \]
\[ M_{11} b_2 + M_{21} = \bar{a}_{21} \quad \Rightarrow \quad M_{21} = \bar{a}_{21} - b_2 \]
\[ M_{11} b_3 + 3M_{21} b_2 + M_{31} = \bar{a}_{31} \]
\[ b_3 + 3(\bar{a}_{21} - b_2)b_2 + M_{31} = \bar{a}_{31} \]
\[ \Rightarrow M_{31} = \bar{a}_{31} - b_3 - 3\bar{a}_{21}b_2 + 3b_2^2 . \]
The terms correspond to \( \beta \bar{g}^2 \) and \( \beta^2 \bar{g}^2 \) are
\[ \frac{\bar{g}^2}{2!} [ M_{12} \beta + M_{22} \frac{\beta^2}{2} ] \]
\[ = \frac{\bar{g}^2}{2!} [ M_{12} \bar{g} + M_{12} \frac{1}{2} b_2 \bar{g}^2 + M_{22} \frac{1}{2} \bar{g}^2 ] \]
\[ = \frac{\bar{g}^2}{2!} [ M_{12} \bar{g} + (M_{12} b_2 + M_{22}) \frac{\bar{g}^2}{2!} ] . \]
Hence
\[ M_{12} = \bar{a}_{12} \]
\[ M_{12} b_2 + M_{22} = \bar{a}_{22} \quad \Rightarrow \quad M_{22} = \bar{a}_{22} - \bar{a}_{12} b_2 . \]
And the term corresponds to \( \beta \bar{g}^3 \) is
\[ \frac{\bar{g}^3}{6} [ M_{13} \beta ] = \frac{\bar{g}^3}{3!} M_{13} \bar{g} \]
\[ \Rightarrow M_{13} = \bar{a}_{13} . \]
Collecting all the \( M_{ij} \), we have
\[ M_{00} = \bar{a}_{00} \]
\[ M_{01} = \bar{a}_{01} \]
\[ M_{02} = \bar{a}_{02} \]
\[ M_{03} = \bar{a}_{03} \]
\[ M_{04} = \bar{a}_{04} \]
\[ M_{10} = 0 \]
\[ M_{11} = 1 \]
\[ M_{12} = \bar{a}_{12} \]
\[ M_{13} = \bar{a}_{13} \]
17
\[ M_{20} = -1 \]
\[ M_{21} = \bar{a}_{21} - b_2 \]
\[ M_{22} = \bar{a}_{22} - \bar{a}_{12}b_2 \]
\[ M_{30} = \bar{a}_{30} + 3b_2 \]
\[ M_{31} = \bar{a}_{31} - b_3 - 3\bar{a}_{21}b_2 + 3b_2^2 \]
\[ M_{40} = \bar{a}_{40} + 4b_3 - 6\bar{a}_{30}b_2 - 15b_2^2. \]

Since, for the location type re-expression, \( M_{30} = -M_{21} \) and \( M_{40} = -M_{31} \), we have
\[ M_{30} = -M_{21} \Rightarrow \bar{a}_{30} + 3b_2 = -\bar{a}_{21} + b_2 \]
\[ \Rightarrow 2b_2 = -(\bar{a}_{30} + \bar{a}_{21}) \]
\[ \Rightarrow b_2 = -\frac{1}{2}(\bar{a}_{30} + \bar{a}_{21}). \]

\[ M_{40} = -M_{31} \]
\[ \Rightarrow \bar{a}_{40} + 4b_3 - 6\bar{a}_{30}b_2 - 15b_2^2 = -\bar{a}_{31} + b_3 + 3\bar{a}_{21}b_2 - 3b_2^2 \]
\[ 3b_3 = -\bar{a}_{31} - \bar{a}_{40} + 3\bar{a}_{21}b_2 + 6\bar{a}_{30}b_2 + 12b_2^2 \]
\[ = -\bar{a}_{31} - \bar{a}_{40} + 3\bar{a}_{21}\left(-\frac{1}{2}(\bar{a}_{30} + \bar{a}_{21})\right) - 3\bar{a}_{30}(\bar{a}_{30} + \bar{a}_{21}) + 3(\bar{a}_{30} + \bar{a}_{21})^2 \]
\[ 6b_3 = -2\bar{a}_{31} - 2\bar{a}_{40} - 3\bar{a}_{21}(\bar{a}_{30} + \bar{a}_{21}) - 6\bar{a}_{30}(\bar{a}_{30} + \bar{a}_{21}) + 6(\bar{a}_{30} + \bar{a}_{21})^2 \]
\[ = -2\bar{a}_{31} - 2\bar{a}_{40} - 3\bar{a}_{21}\bar{a}_{30} - 3\bar{a}_{21}^2 - 6\bar{a}_{30}^2 - 6\bar{a}_{30}\bar{a}_{21} + 6\bar{a}_{30}^2 + 12\bar{a}_{30}\bar{a}_{21} + 6\bar{a}_{21}^2 \]
\[ = -2\bar{a}_{31} - 2\bar{a}_{40} + 3\bar{a}_{21}\bar{a}_{30} + 3\bar{a}_{21}^2 \]
\[ \Rightarrow b_3 = \frac{1}{3}(3\bar{a}_{21}\bar{a}_{30} + 3\bar{a}_{21}^2 - 2\bar{a}_{31} - 2\bar{a}_{40}). \]

Let
\[ u = \bar{y} = d_2 \frac{\bar{y}^2}{2!} + d_3 \frac{\bar{y}^3}{3!} \]
be the new variable where \( d_2 \) and \( d_3 \) are \( O(n^{-1/2}) \) and \( O(n^{-1}) \) respectively. Excluding all \( O(n^{-3/2}) \) or higher terms, we have
\[ u^2 = \bar{y}^2 + d_2 \bar{y}^3 + \left(d_3 \frac{3}{3} + \frac{d_2^2}{4}\right) \bar{y}^4 \]
\[ u^3 = \bar{y}^3 + \frac{3}{2}d_2 \bar{y}^4 \]
\[ u^4 = \bar{y}^4. \]
Also \( du = (1 + d_2 \bar{y} + \frac{d_3}{2} \bar{y}^2) d\bar{y} \) and as in the exponential type re-expression

\[
\ell(\beta; \bar{y}) = \ell(\beta; u) + d_2 \bar{y} + \left(\frac{d_3}{2} - \frac{d_2^2}{2}\right) \bar{y}^2 .
\] (11)

Now expand \( \ell(\beta; u) \) around \((\beta^0, u^0) = (0, 0)\)

\[
\ell(\beta; u) = B_{00} + B_{01} u + B_{02} \frac{u^2}{2!} + B_{03} \frac{u^3}{3!} + B_{04} \frac{u^4}{4!} \\
+ B_{10} \beta + B_{11} \frac{\beta u}{1!} + B_{12} \frac{\beta^2 u^2}{1! 2!} + B_{13} \frac{\beta^3 u^3}{1! 3!} \\
+ B_{20} \frac{\beta^2}{2!} + B_{21} \frac{\beta^2 u}{2! 1!} + B_{22} \frac{\beta^2 u^2}{2! 2!} \\
+ B_{30} \frac{\beta^3}{3!} + B_{31} \frac{\beta^3 u}{3!} \\
+ B_{40} \frac{\beta^4}{4!} + \cdots .
\]

Comparing (10) and (11), the coefficient of

constant term: \( B_{00} = M_{00} = \bar{a}_{00} \)

\[
\frac{\beta}{1!} \text{ term: } B_{10} = M_{10} = 0
\]

\[
\frac{\beta^2}{2!} \text{ term: } B_{20} = M_{20} = -1
\]

\[
\frac{\beta^3}{3!} \text{ term: } B_{30} = M_{30} = \bar{a}_{30} + 3b_2
\]

\[
\frac{\beta^4}{4!} \text{ term: } B_{40} = M_{40} = \bar{a}_{40} + 4b_3 - 6\bar{a}_{30}b_2 - 15b_2^2 .
\]
The terms correspond to \( u \), \( u^2 \), \( u^3 \) and \( u^4 \) are

\[
B_{01}u + B_{02}\frac{u^2}{2} + B_{03}\frac{u^3}{6} + B_{04}\frac{u^4}{24} + d_2\bar{y} + \left(\frac{d_3}{2} - \frac{d_2^2}{2}\right)\bar{y}^2
\]

\[
= d_2\bar{y} + (d_3 - d_2^2)\frac{1}{2}\bar{y}^2
+ B_{01}\bar{y} + B_{01}d_2\frac{1}{2}\bar{y}^2 + B_{01}d_3\frac{1}{6}\bar{y}^3
\]

\[
+ B_{02}\frac{1}{2}\bar{y}^2 + B_{02}\frac{1}{2}d_2\bar{y}^3
+ B_{02}\frac{1}{2}\frac{d_3}{3} + \frac{d_2^2}{4}\bar{y}^4
\]

\[
+ B_{03}\frac{1}{6}\bar{y}^3 + B_{03}\frac{1}{3}d_2\bar{y}^4
+ B_{04}\frac{1}{24}\bar{y}^4
\]

\[
= (B_{01} + d_2)\bar{y} + (d_3 - d_2^2 + B_{01}d_2 + B_{02})\frac{\bar{y}^2}{2!}
+ (B_{01}d_3 + 3B_{02}d_2 + B_{03})\frac{\bar{y}^3}{3!}
\]

\[
+ \left(12B_{02}\left(\frac{d_3}{3} + \frac{d_2^2}{4}\right) + 6B_{03}d_2 + B_{04}\right)\frac{\bar{y}^4}{4!}
\]

Therefore,

\[
B_{01} + d_2 = M_{01} \Rightarrow B_{01} = \bar{a}_{01} - d_2
\]

\[
d_3 - d_2^2 + B_{01}d_2 + B_{02} = M_{02}
\]

\[
\Rightarrow B_{02} = \bar{a}_{02} - d_3 + d_2^2 - B_{01}d_2
\]

\[
B_{01}d_3 + 3B_{02}d_2 + B_{03} = M_{03}
\]

\[
\Rightarrow B_{03} = \bar{a}_{03} - B_{01}d_3 - 3B_{02}d_2
\]

\[
4B_{02}d_3 + 3B_{02}d_2^2 + 6B_{03}d_2 + B_{04} = M_{04}
\]

\[
\Rightarrow B_{04} = \bar{a}_{04} - 4B_{02}d_3 - 3B_{02}d_2^2 - 6B_{03}d_2
\]

The terms correspond to \( \beta u \), \( \beta u^2 \) and \( \beta u^3 \) are

\[
\frac{\beta}{1!}(B_{11}u + B_{12}\frac{1}{2}u^2 + B_{13}\frac{1}{6}u^2)
\]

\[
= \frac{\beta}{1!}\left[B_{11}\bar{y} + B_{11}\frac{1}{2}d_2\bar{y}^2 + B_{11}\frac{1}{6}d_3\bar{y}^3
\right.
\]

\[
+ B_{12}\frac{1}{2}\bar{y}^2 + B_{12}\frac{1}{2}d_2\bar{y}^3
\]

\[
+ \left.B_{13}\frac{1}{6}\bar{y}^3\right]
\]

\[
= \frac{\beta}{1!}\left[B_{11}\bar{y} + (B_{11}d_2 + B_{12})\frac{\bar{y}^2}{2!} + (B_{11}d_3 + 3B_{12}d_2 + B_{13})\frac{\bar{y}^3}{3!}\right].
\]

20
Hence
\[B_{11} = M_{11} = 1\]
\[B_{11}d_2 + B_{12} = M_{12} \implies B_{12} = \bar{a}_{12} - d_2\]
\[B_{11}d_3 + 3B_{12}d_2 + B_{13} = M_{13}\]
\[\implies B_{13} = \bar{a}_{13} - d_3 - 3B_{12}d_2.\]

The terms correspond to $\beta^2 u$ and $\beta^2 u^2$ are
\[
\frac{\beta^2}{2!}(B_{21}u + B_{22}\frac{1}{2}u^2)
\]
\[= \frac{\beta^2}{2!}(B_{21}\tilde{y} + B_{21}d_2\frac{1}{2}\tilde{y}^2 + B_{22}\frac{1}{2}\tilde{y}^2)\]
\[= \frac{\beta^2}{2!}[B_{21}\tilde{y} + (B_{21}d_2 + B_{22})\frac{\tilde{y}^2}{2!}].\]

Therefore
\[B_{21} = M_{21} = \bar{a}_{21} - b_2\]
\[B_{21}d_2 + B_{22} = M_{22} \implies B_{22} = \bar{a}_{22} - \bar{a}_{12}b_2 - B_{21}d_2.\]

Note: $B_{12} = \bar{a}_{12} - d_2 \implies B_{22} = \bar{a}_{22} - b_2d_2 - B_{12}b_2 - B_{21}d_2$ which is the equation in one of the earlier paper.

The term corresponds to $\beta^3 u$ is
\[
\frac{\beta^3}{3!}B_{31}u = \frac{\beta^3}{3!}B_{31}\tilde{y} + \cdots
\]
\[\implies B_{31} = M_{31} = \bar{a}_{31} - b_3 - 3\bar{a}_{21}b_2 + 3b_2^2\]

Note:
\[B_{31} = \bar{a}_{31} - b_3 - 3\bar{a}_{31} - 3\bar{a}_{21}b_2 + 3b_2^2\]
\[= \bar{a}_{31} - b_3 - 3\bar{a}_{21}(\bar{a}_{21} - b_2)\]
\[= \bar{a}_{31} - b_3 - 3B_{21}b_2\]

which is the equation in one of the earlier paper.
Collecting all the $B_{ij}$, we have

\[ B_{00} = \bar{a}_{00} \]
\[ B_{01} = \bar{a}_{01} - d_2 \]
\[ B_{02} = \bar{a}_{02} - d_3 + d_2^2 - B_{01}d_2 \]
\[ B_{03} = \bar{a}_{03} - B_{01}d_3 - 3B_{02}d_2 \]
\[ B_{04} = \bar{a}_{04} - 4B_{02}d_3 - 3B_{02}d_2^2 - 6B_{03}d_2 \]
\[ B_{10} = 0 \]
\[ B_{11} = 1 \]
\[ B_{12} = \bar{a}_{12} - d_2 \]
\[ B_{13} = \bar{a}_{13} - d_3 - 3B_{12}d_2 \]
\[ B_{20} = -1 \]
\[ B_{21} = \bar{a}_{21} - b_2 \]
\[ B_{22} = \bar{a}_{22} - \bar{a}_{12}b_2 - B_{21}d_2 \]
\[ B_{30} = \bar{a}_{30} + 3b_2 \]
\[ B_{31} = \bar{a}_{31} - b_3 - 3\bar{a}_{21}b_2 + 3b_2^2 \]
\[ B_{40} = \bar{a}_{40} + 4b_3 - 6\bar{a}_{30}b_2 - 15b_2^2 \]

where

\[ b_2 = -\frac{1}{2}(\bar{a}_{30} + \bar{a}_{21}) \]
\[ b_3 = \frac{1}{6}(3\bar{a}_{21}\bar{a}_{30} + 3\bar{a}_{21}^2 - 2\bar{a}_{31} - 2\bar{a}_{40}) . \]

Since for the location type re-expression,

\[ B_{30} = -B_{21} = B_{12} \]
\[ B_{40} = -B_{31} = -B_{13} . \]
Thus

\[ B_{12} = B_{30} \Rightarrow \bar{a}_{12} - d_2 = \bar{a}_{30} + 3b_2 \]

\[ \Rightarrow d_2 = \bar{a}_{12} - \bar{a}_{30} - 3b_2 \]

\[ B_{13} = -B_{40} \Rightarrow \bar{a}_{13} - d_3 - 3B_{12}d_2 = -B_{40} \]

\[ \Rightarrow d_3 = \bar{a}_{13} - 3B_{12}d_2 + B_{40} . \]

In summary, for the location type re-expression, we have the following order of calculations for the \( b_2 \), \( b_3 \), \( d_2 \), \( d_3 \) and \( B_{ij} \):

1/ \( b_2 = -\frac{1}{2}(\bar{a}_{30} + \bar{a}_{21}) \).

2/ \( b_3 = \frac{1}{8}(3\bar{a}_{21}\bar{a}_{30} + 3\bar{a}_{21}^2 - 2\bar{a}_{31} - 2\bar{a}_{40}) \).

3/ \( d_2 = \bar{a}_{12} - \bar{a}_{30} - 3b_2 \).

4/ \( B_{00} = \bar{a}_{00} \).

5/ \( B_{10} = 0 \).

6/ \( B_{20} = -1 \).

7/ \( B_{30} = \bar{a}_{30} + 3b_2 \).

8/ \( B_{40} = \bar{a}_{40} + 4b_3 - 6\bar{a}_{30}b_2 - 15b_2^2 \).

9/ \( B_{01} = \bar{a}_{01} - d_2 \).

10/ \( B_{11} = 1 \).

11/ \( B_{21} = \bar{a}_{21} - b_2 = -B_{30} \).

12/ \( B_{31} = \bar{a}_{40} + 4b_3 - 6\bar{a}_{30}b_2 - 15b_2^2 = -B_{40} \).

13/ \( B_{12} = B_{30} \).

14/ \( d_3 = \bar{a}_{13} - 3B_{12}d_2 + B_{40} \).

15/ \( B_{02} = \bar{a}_{02} - d_3 + d_2^2 - B_{01}d_2 \).

16/ \( B_{22} = \bar{a}_{22} - \bar{a}_{12}b_2 - B_{21}d_2 \).

17/ \( B_{03} = \bar{a}_{03} - B_{01}d_3 - 3B_{02}d^2 \).

18/ \( B_{13} = \bar{a}_{13} - d_3 - 3B_{12}d_2 = B_{31} \).

19/ \( B_{04} = \bar{a}_{04} - 4B_{02}d_3 - 3B_{02}d_2^2 - 6B_{03}d_2 \).
Hence we have equations (3.10a), (3.10b) and (3.11) of the paper.

5. Reference