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CURVATURE MEASURES FOR STATISTICAL MODELS: THIRD ORDER MEASURES OF DEPARTURE FROM EXPONENTIAL OR LOCATION FORM

D.A.S. Fraser and N. Reid
Department of Statistics
University of Toronto
Toronto, Canada, M5S 1A1

A. Wong, J. Wu and X. Yuan
Department of Mathematics and Statistics
York University
North York, Canada, M3J 1P3

SUMMARY

The notion of the curvature of a statistical model was extensively developed by Efron (1975); this curvature measures departure from some nearest exponential model and can be viewed as an expectation-type measure of departure. We examine an observed-type measure of departure which includes a sign not present with the Efron measure. We also consider curvature with respect to location models; such models exhibit a fundamental linearity which is quite different from that found with exponential models. The connection between the location and exponential measures of departure is discussed.

Some Keywords: Asymptotics; Curvature; Exponential model; Location model

1. INTRODUCTION

Consider a statistical model $f(y;\theta)$ with a scalar parameter θ. Efron (1975) investigates the curvature of such a model in relationship to the exponential model form

$$g(y;\theta) = \exp\{\varphi(\theta)s(y)\}h(y)$$

with a scalar canonical parameter. The curvature $\gamma(\theta)$ at a parameter value θ is defined by

$$\gamma^2(\theta) = \frac{|M(\theta)|}{i^3(\theta)} = \left\{\frac{\nu_{02}(\theta)}{i^2(\theta)} - \frac{\nu_{11}^2(\theta)}{i^3(\theta)}\right\} \tag{1.1}$$

where

$$M(\theta) = \begin{pmatrix} \nu_{20}(\theta) & \nu_{11}(\theta) \\ \nu_{11}(\theta) & \nu_{02}(\theta) \end{pmatrix}$$

is the covariance matrix of the first $\ell_\theta(\theta;y)$ and second $\ell_{\theta\theta}(\theta;y)$ score variables; the subscript θ denotes differentiation with respect to θ, and $i(\theta) = \mathrm{var}\{\ell_\theta(\theta;y);\theta\} = \nu_{20}(\theta)$ is the expected information.

Hinkley (1980) recorded a modified formula

$$r^2(\theta) = \frac{1}{i^2(\theta)}\left\{\mathrm{var}\{\ell_{\theta\theta}(\theta;y);\theta\} - \frac{\mathrm{cov}^2\{\ell_{\theta\theta}(\theta;y),\ell_\theta(\theta;y);\theta\}}{\mathrm{var}\{\ell_\theta(\theta;y);\theta\}}\right\} \tag{1.2}$$

from which we note easily that $\gamma^2(\theta)$ is the information standardized variance of the residual of $\ell_{\theta\theta}$ regressed in ℓ_θ; this residual is of course zero for an exponential model.

The present discussion of statistical curvature is based on the sample space from of asymptotic statistical models. For this consider a model $f(y;\theta)$ with scalar variable and scalar parameter, and assume that the model has asymptotic properties as some characteristic such as sample size becomes large. We expand the log density $\ell(\theta;y) = \log f(y;\theta)$ in a Taylor series about $(\hat\theta^0, y^0)$ where y^0 is a sample point of interest and $\hat\theta^0 = \hat\theta(y^0)$ is the corresponding maximum likelihood value:

$$\ell(\theta;y) = \Sigma a_{ij}(\theta-\hat\theta^0)^i(y-y^0)^j/i!j!$$

where

$$a_{ij} = \frac{\partial^{i+j}}{\partial\theta^i \partial y^j} \ell(\theta; y)\Big|_{(\hat{\theta}^0, y^0)} . \tag{1.3}$$

It is convenient to examine this expansion by recording just the coefficient matrix $(a_{ij} : i, j = 0, 1, \ldots)$, where we have $a_{10} = 0$ because the expansion is about the maximum likelihood value.

To see the asymptotic form more clearly, we standardize the parameter with respect to observed information,

$$(\theta - \hat{\theta}^0)(-a_{20})^{-1/2} , \tag{1.4}$$

and we standardize the variable with respect to the $(1,1)$ coefficient in the Taylor array,

$$(y - y^0) a_{11}^{-1} (-a_{20})^{1/2} . \tag{1.5}$$

Then letting θ and y now designate these new standardized versions we obtain the Taylor coefficient array

$$\begin{bmatrix} A_{00} & A_{01} & A_{02} & A_{03} & A_{04} \\ 0 & 1 & A_{12} & A_{13} & - \\ -1 & A_{21} & A_{22} & - & - \\ A_{30} & A_{31} & - & - & - \\ A_{40} & - & - & - & - \end{bmatrix} \tag{1.6}$$

where missing elements are $O(n^{-3/2})$ and other elements are $O(n^{1-(i+j)/2})$ with

$$A_{ij} = a_{ij}(-a_{20})^{-(i-j)/2} a_{11}^{-j} \tag{1.7}$$

for $i + j \geq 1$.

The location models and the exponential models form two broad classes each with rather special properties for statistical inference. The exponential models have a linearity that manifests itself in the logarithm of the statistical model and leads to the sufficiency reduction methods by Fisher (1934). By constrast the location models have a linearity that relates the variable and the parameter and manifests itself on the sample space, and then leads to the conditional analysis methods initiated by Fisher (1934). The second type of reduction needed as a primary step in the analysis of a general asymptotic model (Fraser & Reid, 1995). In this general framework the first type of model then represents a singular case but has importance in providing a patter for obtaining approximate significance levels for scalar component parameters. From an overall viewpoint it seems that the conditional reduction of the location models is more important and that the emphasis on sufficiency has been misleading in the overall development of inference methods.

The general asymptotic model described above can be approximated by an exponential model or by a location model (Cakmak et al, 1995); this is accomplished by a change of parameter and a change of variable. Such a change of parameter and change of variable when reexpressed in terms of the location scale standardized versions take the form

$$\begin{gathered} \theta + b_1\theta^2/2n^{1/2} + b_2\theta^3/6n \\ y + c_1 y^2/2n^{1/2} + c_2 y^3/6n \end{gathered} \tag{1.8}$$

to the order $O(n^{-3/2})$.

By appropriate choice of these reexpressions the form (1.6) can be adjusted to almost exponential model form, giving the Taylor array

$$\begin{pmatrix} a + \frac{3\alpha_4 - 5\alpha_3^2 - 12c}{24n} & -\frac{\alpha_3}{2n^{1/2}} & -1 - \frac{\alpha_4 - 2\alpha_3^2 - 5c}{2n} & \frac{\alpha_3}{n^{1/2}} & \frac{\alpha_4 - 3\alpha_3^2 - 6c}{n} \\ 0 & 1 & 0 & 0 & - \\ -1 & 0 & \frac{c}{n} & - & - \\ -\frac{\alpha_3}{n^{1/2}} & 0 & - & - & - \\ -\frac{\alpha_4}{n} & - & - & - & - \end{pmatrix} \tag{1.9}$$

where $a = -(1/2)\log(2\pi)$. If $c = 0$, this corresponds to an exponential model; otherwise, c/n measures departure of the general model from exponential form. We call it the empirical exponential curvature and show in the Appendix that it is an invariant of the model at the data point y^0 or at the maximum likelihood point $\hat{\theta}^0$.

In a parallel manner we can reexpress the variable and parameter by appropriate choice of (1.8) and obtain almost location form:

$$\begin{pmatrix} a + \frac{(3\beta_4 - 5\beta_3^2 - 12C)}{24n} & 0 & -\left\{1 - \frac{5C}{2n}\right\} & \frac{\beta_3}{n^{1/2}} & \frac{(-\beta_4 - 6C)}{n} \\ 0 & 1 & -\frac{\beta_3}{n^{1/2}} & \frac{\beta_4}{n} & - \\ -1 & \frac{\beta_3}{n^{1/2}} & \frac{(-\beta_4 + C)}{n} & - & - \\ -\frac{\beta_3}{n^{1/2}} & \frac{\beta_4}{n} & - & - & - \\ -\frac{\beta_4}{n} & - & - & - & - \end{pmatrix} \tag{1.10}$$

If $C = 0$, this corresponds to a location model; otherwise C/n measures departure of the general model from location form. We call C/n the empirical location curvature and show in the Appendix that it is an invariant of the model at the data point y^0 or at the maximum likelihood value $\hat{\theta}^0$.

For assessing closeness to exponential model form we now have the exceptional measure $\gamma(\theta)$ developed by Efron and we have the observed measure $c(y^0)/n$ discussed above. In Section 2 we obtain a direct expression for the observed measure and the determine the connection between the measures. Some Examples are given.

In Section 3 we obtain direct expressions in the location measure C/n and obtain connections with the exponential measure.

2. DEPARTURE FROM EXPONENTIAL MODEL FORM

Consider an asymptotic model $f(y;\theta)$ having scalar variable y and scalar parameter θ. Under appropriate change of variable and change of parameter we obtain the exponential type array (1.9). If $c = 0$ the model is exponential to third order. Otherwise, the model has exponential characteristics at y^0 and to first derivative about y^0 and the nonexponential characteristic c/n shows with respect to second derivatives at y^0.

The empirical exponential curvature c/n can be reexpressed in terms of the initial Taylor coefficients $a_{ij} = (\partial^{i+j}/\partial\theta^i\partial y^j)\ell(\theta;y)\Big|_{(\hat{\theta}^0,y^0)}$ giving

$$\frac{c}{n} = \frac{1}{a_{11}^3}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \frac{1}{a_{11}^2}(a_{22} - \frac{a_{12}a_{21}}{a_{11}}) \ . \tag{2.1}$$

We now use the canonical expansion given by (1.9) to determine the connection between the empirical curvature c/n and the expected curvature $\gamma(\theta)$, both taken at the parameter value $\theta = 0$ of the centered model. From (1.9) we obtain

$$\begin{aligned} \ell_\theta(\theta;y) &= y \\ \ell_{\theta\theta}(\theta;y) &= -1 + \frac{c}{n}\frac{y^2}{2} \ . \end{aligned} \tag{2.2}$$

Also with the canonical model we have $\text{var}(y;0) = 1$, $\text{cov}(y, y^2; 0) = 0$ to first order and thus $\text{cov}(\ell_\theta, \ell_{\theta\theta}; \theta) = 0$ to third order. It follows that the residual of $\ell_{\theta\theta}$ regressed on ℓ_θ at $\theta = 0$ is

$$\text{Res}(\theta; y) = -1 + \frac{c}{n}\frac{y^2}{2} + O(n^{-3/2})$$

with standard deviation

$$\text{SD}\{\text{Res}(\theta; y); 0\} = \frac{|c|}{n}\frac{1}{\sqrt{2}} \ . \tag{2.3}$$

Thus, in terms of the original model, we obtain

$$\gamma(0) = \frac{|c|}{n}\frac{1}{\sqrt{2}} + O(n^{-3/2}) \ . \tag{2.4}$$

It follows that the absolute value of the empirical curvature c/n to the third order is equal to $\sqrt{2}$ times the Efron curvature . The sign of the empirical curvature indicates an essential property of the model as can be seen from the following form for the canonical model (1.9)

$$f(y;\theta) = \exp\{y\theta + \frac{c}{n}\frac{y^2\theta^2}{4} - \gamma(\theta)\} f(y) \ . \tag{2.5}$$

Example 1. Consider a sample $(y_1, \ldots, y_n)$ from the model $f(y;\theta) = \theta^{-1}\exp\{-y/\theta\}$ on $(0, \infty)$. Then $\ell(\theta; y) = -n\log\theta - n\bar{y}/\theta$ which depends on the scalar sufficient statistic $\bar{y}$. We easily obtain $\gamma(\theta) = 0$ and at any chosen $\bar{y}^0$ obtain $c/n = 0$. This is of course in accord with the model being exponential.

Example 2. Consider the statistical model

$$f(y;\theta) = \frac{(1 - d\theta^2/2)^{1/2}}{(2\pi)^{1/2}}\exp\{-1/2(y-\theta)^2 + \frac{1}{4}d\theta^2(y-\theta)^2\} \tag{2.6}$$

The Taylor series about the $(\theta; y) = (0, 0)$ gives the array

$$\begin{pmatrix} a & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & - \\ -1-\frac{d}{2} & 0 & d & - & - \\ 0 & -3d & - & - & - \\ 6d - \frac{3}{2}d^2 & - & - & - & - \end{pmatrix} \tag{2.7}$$

from which we obtain the empirical curvature $c/n = d$. The first and second score variables at $\theta = 0$ are $\ell_\theta = y$, $\ell_{\theta\theta} = -1 - d/2 + dy^2/2$ with covariance matrix

$$M(0) = \begin{bmatrix} 1 & 0 \\ 0 & d^2/2 \end{bmatrix} , \tag{2.8}$$

Also we note that positive and negative values for the empirical curvature give quite different model structure near $\theta = 0$.

Example 3. Consider a location model $f(y-\theta) = \exp\{g(y-\theta)\}$ and suppose that the model has been centered so $\hat{\theta} = y$. The empirical exponential curvature is then obtained as

$$c = \frac{g^{(4)}(0) - \{g^{(3)}(0)\}^2/g^{(2)}(0)}{\{g^{(2)}(0)\}^2}$$

where $g^{(i)}(0)$ is the ith derivative of $g(z)$ at $z = 0$. For the particular case of the Cauchy we have $g^{(2)}(0) = -2$, $g^{(3)}(0) = 0$, $g^{(4)}(0) = 12$ giving the empirical curvature $c = 3$. With somewhat more calculation we obtain the Efron curvature $\gamma(0) = 3/\sqrt{2}$.

3. DEPARTURE FROM LOCATION MODEL FORM

Consider an asymptotic model $f(y;\theta)$ with scalar variable y and scalar parameter θ. Under appropriate change of variable and change of parameter we obtain the location type array (1.10) in which C/n records departure from the location model structure indicated by the first two columns.

The empirical exponential curvature can be reexpressed in terms of the coefficients A_{ij} given in the centered array (1.6)

$$\frac{C}{n} = A_{22} - A_{12}A_{21} + 1/2(A_{30}^2 - A_{21}^2) + A_{30}A_{21} + \frac{1}{3}(4A_{31} + A_{40}) \tag{3.1}$$

or in terms of the original coefficients by using (1.7)

$$\begin{aligned} \frac{C}{n} = \frac{a_{22}a_{11} - a_{12}a_{21}}{a_{11}^2} &+ \frac{1}{2}\{(-a_{20})^{-3}a_{30}^2 - (-a_{20})^{-1}a_{11}^{-2}a_{21}\} + (-a_{20})^{-2}a_{11}^{-1}a_{30}a_{21} \\ &+ \frac{1}{3}\{4(-a_{20})^{-1}a_{11}^{-1}a_{31} + (-a_{20})^{-2}a_{40}\} \end{aligned} \tag{3.2}$$

Example 3 continued Consider a single observation from the centered location model $f(y-\theta)=\exp\{g(y-\theta)\}$. We then obtain using (1.7)

$$a_{ij}=(-1)^i g^{(i+j)}(\theta)\ ,\ 1,j\geq 0$$

$$A_{ij}=(-1)^i\{-g^{(2)}(0)\}^{-(i-j)/2}\{-g^{(2)}(0)\}^{-j}g^{(i+j)}(0)\ ,\ i+j\geq 1\ .$$

It follows that

$$A_{12}=\{-g^{(2)}(0)\}^{-3/2}g^{(3)}(0)=-A_{21}=-A_{30}$$

$$A_{22}=\{-g^{(2)}(0)\}^{-2}g^{(4)}(0)=-A_{31}=A_{40}$$

which gives $C=0$. Thus the empirical location curvature of a location model is 0.

From the general expression (3.1) for the empirical location curvature.

$$\frac{C}{n}=\frac{c}{n}+\frac{c_2}{n} \tag{3.4}$$

where c/n is the empirical exponential curvature and

$$\frac{c_2}{n}=1/2(A_{30}^2-A_{21}^2)+A_{30}A_{21}+\frac{1}{3}(4A_{31}+A_{40}) \tag{3.5}$$

measures the change from exponential to location curvature. We note that c_2/n involves only elements in the first two columns of (1.6).

Tangent exponential and tangent location models are obtained by setting c/n and C/n equal to zero in (1.9) and (1.10). In contrast to the tangent location model the tangent exponential model at a data point y^0 has a relatively simple third order expression in terms of likelihood and likelihood gradient at the data point,

$$\frac{k}{(2\pi)^{1/2}}\exp\{\ell^0(\theta)-\ell^0(\hat{\theta}^0)+(\varphi-\hat{\varphi}^0)s\}|\tilde{\jmath}_{\varphi\varphi}|^{-1/2}ds\ ; \tag{3.6}$$

for this, $\ell^0(\theta)=\ell(\theta;y^0)$, is the observed likelihood $\varphi=\partial\ell(\theta;y)\partial y\big|_{y^0}$ is likelihood gradient giving a nominal reparameterizations, $s=\partial\ell(\theta;y)/\partial\varphi\big|_{\hat{\theta}}$ is the corresponding score variable, and $\tilde{\jmath}_{\varphi\varphi}$ is the observed information from the tilted likelihood in the exponent.

Now suppose we calculate the exponential to location adjustment (3.4) from an exponential approximation (1.9); we obtain

$$\frac{c_2}{n}=\frac{1}{2}\frac{\alpha_3^2}{n}+\frac{1}{3}\frac{\alpha_4}{n}\ ,$$

agreeing with relations obtained in Cakmak et al. (1995).

Example 1 continued. Consider the sample from the simple exponential model and examine the reduced model for $\bar{y}$ with likelihood $\ell(\theta; y) = -n\log\theta - ny/\theta)$ where we use y in place of $\bar{y}$. We have seen that the empirical curvature $c/n = 0$. Consider a value y^0 with corresponding $\hat{\theta}^0 = y^0$. We then obtain $A_{30} = 4n^{-1/2}, A_{21} = -2n^{-1/2}$, $A_{31} = 6n^{-1}$, $A_{40} = -18n^{-1}$ giving $c_2/n = 0$ and then $C/n = 0$. We thus obtain location model structure which is of course easily seen directly.

Example 4. Consider the log gamma model $\Gamma^{-1}(\theta)\exp(\theta y - e^y)$. The log likelihood is

$$\ell(\theta; y) = \theta y - e^y - \psi(\theta)$$

where $\psi(\theta) = \log\Gamma(\theta)$. The maximum likelihood value is obtained from $\hat{\theta} - \psi'(\hat{\theta})$. At a point $(y^0, \hat{\theta}^0)$ we then calculate

$$A_{21} = A_{12} = A_{31} = A_{22} = 0$$

$$A_{30} = \{\psi^{(2)}(\hat{\theta}^0)\}^{-3/2}\{-\psi^{(3)}(\hat{\theta}^0)\}\ ,\ A_{40} = \{\psi^{(2)}(\hat{\theta}^0)\}^{-2}\{-\psi^{(4)}(\hat{\theta}^0)\}$$

and then obtain

$$C/n = c_2/n = \frac{1}{2}A_{30}^2 + \frac{1}{3}A_{40} = \frac{1}{4}\{\psi^{(2)}(\hat{\theta}^0)\}^{-3}[\frac{1}{2}\{\psi^{(3)}(\hat{\theta}^0)\}^2 - \frac{1}{3}\psi^{(2)}(\hat{\theta}^0)\psi^{(4)}(\hat{\theta}^0)]$$

APPENDIX

We now verify that the empirical exponential curvature is an invariant of the statistical model at a point $(y^0, \hat{\theta}^0)$ consider

$$c(\theta; y) = \frac{1}{a_{11}^3(\theta; y)}\{a_{11}(\theta; y)a_{22}(\theta; y) - a_{12}(\theta; y)a_{21}(\theta; y)\} \tag{1}$$

where $a_{ij}(\theta; y)$ is given by (1.3) and $c/n = c(\hat{\theta}(y^0; y^0)$. Let $\varphi = \varphi(\theta)$ be monotone twice differentiable transformation with inverse $\theta = \theta(\varphi)$; then

$$\begin{aligned} a_{11} &= \ell_{\varphi;y}\varphi_\theta \\ a_{22} &= \ell_{\varphi\varphi;yy}\varphi_\theta^2 + \ell_{\varphi;yy}\varphi_{\theta\theta} \\ a_{12} &= \ell_{\varphi;yy}\varphi_\theta \\ a_{21} &= \ell_{\varphi\varphi;y}\varphi_\theta^2 + \ell_{\varphi;y}\varphi_{\theta\theta} \end{aligned} \tag{2}$$

in obvious notation with subscripts denoting differentiation. By substituting (2) in (1) we see that $c(\theta; y)$ has the same form in terms or the new φ. In a similar way we have invariance under changed variable. We now verify that the empirical location curvature is an invariant of the statistical model. As we have the verification for c/n, we not then from (3.4) that it suffices to do the verification for c_2/n. Consider functions defined as follows.

$$A_{ij}(\theta; y) = (-a_{20}(\theta; y))^{-(i-j)/2}a_{11}^{-j}(\theta; y)a_{ij}(\theta; y) \tag{3}$$

let $\varphi = \varphi(\theta)$ be a any strictly monotone differentiable parameter transformation. Then with all functions evaluated at $(\hat{\theta}(y^0), y^0))$ we obtain.

$$\begin{aligned} A_{30}^2 &= (-a_{30})^{-3}a_{30}^2 = (-\ell_{\varphi\varphi})^{-3}\varphi_\theta^{-6}[\ell_{\varphi\varphi\varphi}\varphi_\theta^3 + 3\ell_{\varphi\varphi}\varphi_\theta\varphi_{\theta\theta}]^2 \\ &= (-\ell_{\varphi\varphi})^{-3}(\ell_{\varphi\varphi\varphi} + 6\ell_{\varphi\varphi\varphi}\ell_{\varphi\varphi}\varphi_\theta^{-2}\varphi_{\theta\theta} + 9\ell_{\varphi\varphi}^2\varphi_\theta^{-4}\varphi_{\theta\theta}^2) \\ A_{21}^2 &= (-a_{20})^{-1}a_{11}^{-2}a_{21}^2 \\ &= (-\ell_{\varphi\varphi})^{-1}\ell_{\varphi;y}^{-2}\varphi_\theta^{-4}(\ell_{\varphi\varphi;y}\varphi_\theta^2 + \ell_{\varphi;y}\varphi_{\theta\theta})^2 \\ &= (-\ell_{\varphi\varphi})^{-1}\ell_{\varphi;y}^{-2}(\ell_{\varphi\varphi;y}^2 + 2\ell_{\varphi\varphi;y}\ell_{\varphi;y}\varphi_\theta^{-2}\varphi_{\theta\theta} + \ell_{\varphi;y}^2\varphi_\theta^{-4}\varphi_{\theta\theta}^2) \end{aligned}$$

$$
\begin{aligned}
A_{30}A_{21} &= (-a_{20})^{-2}a_{11}^{-1}a_{30}a_{21} \\
&= (-\ell_{\varphi\varphi})^{-2}\ell_{\varphi;y}^{-1}\varphi_\theta^{-5}(\ell_{\varphi\varphi\varphi}\varphi_\theta^3 + 3\ell_{\varphi\varphi}\varphi_\theta\varphi_{\theta\theta})(\ell_{\varphi\varphi;y}\varphi_\theta^2 + \ell_{\varphi;y}\varphi_{\theta\theta}) \\
&= (-\ell_{\varphi\varphi})^{-2}\ell_{\varphi;y}^{-1}(\ell_{\varphi\varphi\varphi}\ell_{\varphi\varphi;y} + \ell_{\varphi\varphi\varphi}\ell_{\varphi;y}\varphi_\theta^{-2}\varphi_{\theta\theta} \\
&\quad + 3\ell_{\varphi\varphi}\ell_{\varphi\varphi;y}\varphi_\theta^{-2}\varphi_{\theta\theta} + 3\ell_{\varphi\varphi}\ell_{\varphi;y}\varphi_\theta^{-4}\varphi_{\theta\theta}^2) \\
A_{31} &= 1 - a_{20})^{-1}a_{11}^{-1}a_{31} \\
&= (-\ell_{\varphi\varphi})^{-1}\ell_{\varphi;y}^{-1}\varphi_\theta^{-3}(\ell_{\varphi\varphi\varphi;y}\varphi_\theta^3 + 3\ell_{\varphi\varphi;y}\varphi_\theta\varphi_{\theta\theta} + \ell_{\varphi;y}\varphi_{\theta\theta\theta}) \\
&= (-\ell_{\varphi\varphi})^{-1}\ell_{\varphi;y}^{-1}(\ell_{\varphi\varphi\varphi;y} + 3\ell_{\varphi\varphi;y}\varphi_\theta^{-2}\varphi_{\theta\theta} + \ell_{\varphi;y}\varphi_\theta^{-3}\varphi_{\theta\theta\theta}) \\
A_{40} &= (-a_{20})^{-2}a_{40} \\
&= (-\ell_{\varphi\varphi})^{-2}(\ell_{\varphi\varphi\varphi\varphi} + 6\ell_{\varphi\varphi\varphi}\varphi_\theta^{-2}\varphi_{\theta\theta} + 3\ell_{\varphi\varphi}\varphi_\theta^{-4}\varphi_{\theta\theta}^2 + 4\ell_{\varphi\varphi}\varphi_\theta^{-3}\varphi_{\theta\theta\theta})
\end{aligned}
$$

Let $B_{ij} = (-b_{20})^{-(i-j)/2}b_{11}^{-j}b_{ij}$ be the expression (3) as determined from the parameterization φ. Then substituting from above we obtain

$$
1/2(A_{30}^2 - A_{21}^2) + A_{30}A_{21} + \frac{1}{3}(4A_{31} + A_{40}) = \frac{1}{2}(B_{30}^2 - B_{21}^2) + B_{30}B_{21} + \frac{1}{3}(4B_{31} + B_{40})
$$

giving the parameterization invariance of c_2/n and of C/n. In the same pattern we also obtain invariance under change of variable.

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