Conical tests: Observed levels of significance and confidence regions

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Tests and confidence regions for a p-parameter nonnormal model can require integration in p dimensions, for example with a Bayesian or structural model. For small p, the computer integrations are manageable, see for example Fraser [3], Naylor and Smith [6], but for p beyond 5, 6, or 7, the integrations become unfeasible. This paper proposes conical tests of significance which involve manageable computer calculations. The conical tests also provide confidence regions, giving the confidence bound at a chosen level of significance in any direction from a central 0-confidence point.

1. INTRODUCTION

Consider the regression model \( y = X \beta + \epsilon \) where \( X \) is an \( n \times p \) matrix of rank \( p \) and \( \epsilon \) is a sample from a specified distribution, (e.g., Student (6)). The model, as given, is an error or structural model (see Fraser [5]) and leads to a conditional analysis in \( p+1 \) dimensions; \( p \) for the regression coefficients \( b_1, \ldots, b_p \) of \( y \) on the columns of \( X \) and 1 for the error standard deviation \( \sigma \). A Bayesian analysis would combine a prior with the likelihood and again a distribution in \( p+1 \) dimensions is obtained, \( p \) for \( \beta_1, \ldots, \beta_p \) and 1 for \( \sigma \).
The analysis has been examined in detail for \( p = 1 \) (Fraser [3]) and for \( p = 2 \) (Fick [1]). For higher dimensions the integration requires greatly increased computer time and capacity and with reinforcing techniques has been extended to \( p = 6 \) (Naylor and Smith [6]). The extensive computer requirements have meant that the conditional analyses are in general not used for routine problems or even for a detailed examination of effects of various non-normal error distributions.

Consider a statistical model whose analysis leads to a vector pivotal quantity \( \mathbf{v} = (v_1, \ldots, v_p) = q(t, \theta, s) \) where \( t, s \) are statistics, \( (\theta_1, \ldots, \theta_p) \) is a \( p \)-dimensional parameter. We assume the density function of \( \mathbf{v} \) is known in functional form \( g(\mathbf{v}) \), the availability or not of the normalizing constant does not matter. We also assume as part of the development argument that the density function has a single maximum; in many problems this is appropriate and corresponds to a similar property for the likelihood. Also we suppose that \( \mathbf{v} \) provides some natural measure of departure of data from that expected. We relocate the pivotal distribution so the maximum of \( g(\mathbf{v}) \) is at \( \hat{\theta} \).

2. **CONICAL TEST**

Consider a test for the hypothesis \( \theta = \theta^0 \). A traditional interpretation of an observed level of significance, abbreviated \( \text{OLS} \), is the probability of a value as far as or further than the observed value is from that expected. The conical test evolves directly from a natural representation of this interpretation.

The central value for the pivotal variable is \( \hat{\theta} \). This is a consequence of our relocation of the pivotal distribution. The observed value of the pivotal variable
under the hypothesis \( \theta = \theta^0 \) is \( \tilde{v} = q(t^0, \theta^0, s^0) \) where \( t^0, s^0 \) are the observed values of \( t, s \).

The probability of a value \( \tilde{v} \) as far as or further than \( v^0 \) from \( v^0 \) is obtained in a cone radiating from \( v^0 \) and is calculated conditionally given the direction \( d \) of \( v^0 \). Let \( r = ||v||, r^0 = ||v^0|| \) and \( d = \frac{v^0}{r^0} \); then the OLS is \( (w = r/r^0) \):

\[
(2.1) \quad p(r^0, d) = \frac{\int_r^\infty r^{p-1} g(rd) dr}{\int_0^\infty r^{p-1} g(rd) dr} = \frac{\int^{\infty} w^{p-1} g(wv^0) dw}{\int_0^{\infty} w^{p-1} g(wv^0) dw}
\]

The logic underlying this is that the direction in which a departure occurs is not of 'departure' interest and it is appropriate then to examine departure from that expected only in the observed direction. More conventionally, we can examine departure from the centre of the pivotal density by using the procedure just described and obtain a \((1-\alpha)\) contour as

\[
\{v: p(r,d) = \alpha\}.
\]

Points \( \tilde{v} \) on such a contour satisfy \( p(\tilde{v}) = \alpha \). Accordingly we define a marginal OLS by calculating \( p(\tilde{v}^0) \). Note then that the earlier conditional OLS is in fact equal to the marginal OLS calculated using the special contours just described, and thus is a global measure of departure from the centre of the distribution. Thus having noted that the conditional OLS can be viewed as a direction insensitive marginal OLS, we revert to our preferred conditional viewpoint for subsequent discussions of the conical significance tests and confidence regions.
3. CONICAL CONFIDENCE

Let us now consider the formation of a \((1-\alpha)\) confidence region for \(\tilde{\theta}\). For this, we assume the relation \(q(t,\tilde{\theta},s) = \gamma\) leads to an accessible one-to-one relation between \(\gamma\) and \(\tilde{\theta}\). In particular, we designate by \(\tilde{\theta}\) the parameter value that corresponds to the maximum density value for \(\gamma\), the value which is \(\gamma = 0\) by our earlier relocation:
\[
q(t,\tilde{\theta},s) = 0.
\]

To obtain the \((1-\alpha)\) confidence boundary in a direction \(d\) from the maximum density point we solve

\[
(3.1) \quad p(r,d) = \alpha
\]

for \(r = r(\alpha,d)\) and then calculate the corresponding \(\tilde{\theta}\) boundary point from
\[
q(t^0,\tilde{\theta},s^0) = r(\alpha,d)d.
\]

In this way, we can determine the \((1-\alpha)\) boundary point in any direction we are interested in. We note that for many examples including the illustrative examples in the next section, the relation of \(\tilde{\theta}\) to \(rd\) is linear and the last equation is easily solved giving a distance from \(\tilde{\theta}\) to the boundary in any chosen direction.

4. SOME EXAMPLES

We now consider some familiar examples to see how the general conical procedure works when the needed integration can be performed analytically.

4.1 Real valued location parameter

Suppose that \(v = t-\theta\) has probability density function \(g(v)\) with maximum by relocation at \(v=0\). This case can
arise from a model \( y = \theta l + e \) where \( e \) is a sample from a distribution with density function \( f(e) \). The error \( e \) can be written

\[
e = y - \theta l = y - \bar{y}l + \bar{y}l - \theta l
\]

\[
= d + (\bar{y} - \theta)l = u + \bar{y} - \theta
\]

where \( d \) is the vector \( y - \bar{y}l = e - \bar{e}l \) and \( u = \bar{y} - \theta \) is a scalar.

The vector \( d = e - \bar{e}l \) is perpendicular to \( l \) and its value is known \( d^0 = e - \bar{e}l \) as soon as \( y^0 \) is known. The pivotal quantity (the conditional error variable with an error or structural model) is \( u = \bar{y} - \theta \) and its distribution is

\[
\bar{g}(u) = \frac{f(u|d)}{k^{-1}(d)f(u|d)} = k^{-1}(d)f(u|d)
\]

\[
= \int_{-\infty}^{+\infty} f(u|d)du
\]

where \( k(d) = \int_{-\infty}^{+\infty} f(u|d)du \) is the marginal density of \( d \) with respect to \( \sqrt{nd}dd \) and \( dd \) designates Euclidian volume in the space \( \mathbb{L}^2(l) \) for \( d \) (see Fraser [4,5]). Let \( u^* \) be the maximizing value of \( \bar{g}(u) \) and write \( v = u - u^* \). The density for \( v = \bar{y} - u^* - \theta = t - \theta \) is then

\[
g(v) = k^{-1}(d)f((v+u^*)l+d)
\]

we do not need directly the value \( k(d) \) although it arises from the calculations; note \( t = \bar{y} - u^* \).

Consider a test for \( \theta = \theta^0 \). If \( t \) is larger than \( \theta^0 \), then
$$\text{OLS} = \int_{t-\theta^0}^{\infty} g(v) dv / \int_{0}^{\infty} g(v) dv$$

and if $t$ is less than $\theta^0$,

$$\text{OLS} = \int_{-\infty}^{t-\theta^0} g(v) dv / \int_{-\infty}^{0} g(v) dv$$

To obtain a $(1-\alpha)$ confidence region, we calculate $v_1$ and $v_2$ such that

$$\int_{-\infty}^{v_1} g(v) dv = \alpha \int_{-\infty}^{0} g(v) dv$$

$$\int_{v_2}^{\infty} g(v) dv = \alpha \int_{0}^{\infty} g(v) dv$$

The $(1-\alpha)$ confidence interval for $\theta$ is $(t-v_2, t-v_1)$; note that $v_1$ is negative.

To give some indication of how this conical confidence region works, suppose that

$$g(v) = \begin{cases} 
-k e^{-v^2/2\sigma^2_1} & \text{for } v < 0 \\
-k e^{-v^2/2\sigma^2_2} & \text{for } v > 0 
\end{cases}$$
that is two normal density functions that join at \( v=0 \); suppose \( \sigma_1 \) is small and \( \sigma_2 \) is large. Then for a 95% confidence interval \( v_1 = -1.96\sigma_1 \), \( v_2 = 1.96\sigma_2 \), and we note that the 95% confidence points are at the usual points for the particular normal tails. For a first comparison consider an equal probability split for the two tails. This would move both \( v_1 \) and \( v_2 \) to the right; indeed, we can note that the constant \( k \) in \( g(v) \) is equal to \( \frac{2}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} \) and the equal probability split has \( v_1 \) and \( v_2 \) satisfying

\[
\int_{-\infty}^{-\frac{v_1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}}} e^{-v^2/2\sigma_1^2} dv = 0.025 = \int_{\frac{v_2}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}}}^{\infty} e^{-v^2/2\sigma_2^2} dv;
\]

(modification needed for extreme ratios \( \sigma_1 \) to \( \sigma_2 \)) and it follows immediately that \( -1.96\sigma_1 < v_1 \) and \( 1.96\sigma_2 < v_2 \). For a second comparison consider the equidensity level test. As the two normal tails are matched at the origin it follows that \( v_1 = -1.96\sigma_1 \), \( v_2 = 1.96\sigma_2 \) thus giving agreement with the conical approach.

The \((1-\alpha)\) confidence interval obtained through an equal probability split between the tails is shifted to the left with respect to the conical \( 1-\alpha \) confidence interval. On the other hand the equi-density confidence interval gives agreement. We note that if the density \( g(v) \) is symmetric, the conical test leads to equal probability \( \alpha/2 \) on each tail of that pivotal density.

4.2 Normal regression model when \( \sigma \) is known

Consider the model \( \tilde{y} = X\beta + \tilde{e} \) where \( \tilde{e} \) is a sample from the Normal\((0,\sigma_0^2)\) distribution. The error \( \tilde{e} \) can be
written \( e = Xb(e) + d(e) = X(b(y) - \beta) + d(y) \) where 
\( b(y) = (X'X)^{-1}X'y \) is the vector of regression coefficients 
of \( y \) on the columns of \( X \) and \( d(y) \) is the residual 
vector. Note that \( d(e) = d(y) \) can have its value cal-
culated from the observed value for \( y = y^0 \). The pivotal 
variable is \( b = b(e) = b(y) - \beta \) and the conditional dis-
tribution of \( b \) given \( d \) is (see [2], p 171; [4], p 480).

\[
g(b) = (2\pi \sigma_0^2)^{-p/2}(X'X)^{1/2}\exp\left\{-\frac{b'X'Xb}{2\sigma_0^2}\right\}
\]

and has maximum at \( b = 0 \). Consider the conical test of 
the parameter \( \beta = \beta^0 \). The corresponding value of the 
pivotal variable is \( b_0 = b(y^0) - \beta^0 \).

Let us compute the OLS given by (2.1). To simplify 
our notation we use an alternative real variable \( w \) re-
placing the variable \( r \) used in (2.1); the variable \( v \) 
is measured in units with respect to the observed point 
\( b^0 \); such that \( b'X'Xb = w^2 b_0'X'Xb_0 \). The observed level 
of significance becomes

\[
\frac{\int_1^\infty w^{p-1}\exp\left\{-\frac{w^2}{2\sigma_0^2} b_0'X'Xb_0\right\}dw}{\int_0^\infty w^{p-1}\exp\left\{-\frac{w^2}{2\sigma_0^2} b_0'X'Xb_0\right\}dw}
\]

Let \( \chi^2 = b_0'X'Xb_0/\sigma_0^2 = w^2 b_0'X'Xb_0/\sigma_0^2 \); then
\[
\text{OLS} = \frac{\int_0^{\infty} (\chi^2)^{(p/2) - 1} \exp\{-\chi^2/2\} \, d(\chi^2)}{\int_0^{\infty} (\chi^2)^{(p/2) - 1} \exp\{-\chi^2/2\} \, d(\chi^2)}
\]
\[
= \int_0^{\infty} h_p(\chi^2) \, d(\chi^2)
\]

where \( h_p(\cdot) \) is the chi-square density with \( p \) degrees of freedom and \( \chi_0^2 = b'\Sigma Xb / \sigma_0^2 \). This coincides with the ordinary chi-square measure for OLS.

In the natural follow through, the conical \((1-\alpha)\) confidence region is

\[
\{ \beta: (\tilde{b}(y^0) - \beta)'X'(\tilde{b}(y^0) - \beta) \leq \sigma_0^2 \chi^2_p(\alpha) \}
\]

which is the ordinary chi-square confidence region.

4.3 Normal regression model when \( \sigma \) is unknown

This case requires the introduction of a slightly different coordinate system and will be examined as an application of the general regression model in the next section.

5. The conical methods for the nonnormal regression model

We now discuss the implementation of conical tests and confidence regions in the context of the nonnormal regression model. All the complexities likely to be encountered seem present with this model although perhaps not to the maximum degree possible.
Let \( y = X\beta + \epsilon \) where \( X \) has full column rank \( p \) and \( \epsilon \) is a sample from a density \( f(\epsilon) \) having a bell-shaped form such as a Student(\( f \)) density or some variant thereof exhibiting tail thickness, skewness, and central humping as appropriate to an application.

5.1 The system of coordinates

As discussed in Fraser [3,4,5], we use least squares coordinates, for their familiarity and convenience of calculation: \( \sim_{y} = (X'X)^{-1}X'y \) are the coordinates of \( y \) on the column vectors of \( X \), \( s_y = \| y - X\bar{b}(y) \|/(n-p)^{1/2} \) is the coordinate with respect to \( \sqrt{n-p} \) times the unit residual vector \( \sim_d = (y - X\bar{b}(y))/\| y - X\bar{b}(y) \| \). We then introduce coordinates \( \sim_t \) and \( s \) which separate \( \beta \) and \( \sigma \):

\[
\sim_t = \frac{\sim_{y} - \beta}{s_y} = \frac{(\sim_{y} - \beta)/\sigma}{s_y/\sigma} = \frac{\sim_{e}}{s_e} = \frac{\sim}{s}; \quad s = \frac{s_y}{\sigma}
\]

Then \( \sim_{e} = \sim_{y} - X\beta = X(\sim_{y} - \beta) + \sim_{y} - X\bar{b}(y) \) can be re-expressed as

\[
\sim_{e} = \sigma X\bar{b} + \sqrt{n-p} \sigma sd
\]

\[
= \sigma [Xst + \sqrt{n-p} sd]
\]

from which the differential can be calculated,

\[
(5.1) \quad d\sim_{e} = (\sqrt{n-p} s)^{n-p-1} \sqrt{n-p} dstdX'X^{1/2} s^{p-1} dt
\]

\[
= (\sqrt{n-p})^{n-p} \sqrt{n-1} s^{n-1} dstdtt
\]

For detailed calculations, see Fraser [4,5]. The joint distribution of \( s \) and \( t \) given \( \sim_d \) is then

\[
k^{-1}(d)f(Xst + \sqrt{n-p} sd)s^{n-1} dsdt
\]
where \( k(d) \) represents the marginal density function of \( d \) up to a scalar multiple. For subsequent calculations of the conical test and confidence regions the norming constant is not required. Accordingly, we take

\[
g_1(t,s) = k f(Xst + \sqrt{n - p} sd)s^{n-1},
\]

adding perhaps a factor to place the expression on a suitable range for the calculations (say to avoid underflow).

5.2 Tests and confidence regions

For conical tests and confidence regions for \( \beta \), we need the marginal density for \( t \). The density of \( t \) is

\[
g_2(t) = \int_0^\infty g_1(t,s)ds
\]

where \( g_1(t,s) \) is given in (5.2).

The computation of \( g_2(t) \) requires a one-dimensional integration and a computer subroutine would be established for this computation.

Note that we are integrating out the effects of the nuisance parameter \( \sigma \). In more complex problems, this integration with respect to nuisance parameters may well be in more than one dimension and the accessibility of the \( g_2(t) \) would correspondingly be effected. Alternatives will be discussed briefly in Section 5.4.

We now have the pivotal or structural equation

\[
t = \frac{b(y) - \beta}{s_y}
\]
with relative density function $g_2(t)$. The general discussion in Section 2 leads to relocating the density function at the origin $\hat{t}$. For this, we would use a steepest ascent or equivalent subroutine to find the point $T$ at which $g_2$ achieves its maximum value. Then, let $T = t - \hat{t}$; $T = (b(y) - s\hat{t} - \hat{\beta})/s\hat{y}$. has relative density $g_3(T) = g_2(T + t)$. (Note that calculating values for $g_2$ at various $T$ values requires a location adjustment to recenter the first subroutine.) The maximum pivotal density estimate of $\beta$ is $\hat{\beta} = b(y) - s\hat{y}$. The procedures in sections 2 and 3 then allow the conical testing of any $\beta$ value or the formation of conical-based confidence regions about $\hat{\beta}$.

To test $\beta = \beta_0$, we calculate $T_0 = (b(y) - s\hat{t} - \hat{\beta}_0)/s\hat{y}$ and then the OLS from formula (2.1) using $g_3(T)$. This value provides the assessment of the hypothesis.

Consider the confidence bound in a direction $-\delta$ from $\hat{\beta}$: say $\beta = \hat{\beta} - \rho\delta$. The pivotal equation is $T = \rho\delta/s\hat{y}$. Let $r(\alpha, \delta)$ be the solution to (3.1): this involves the one-dimensional integration required for (2.1) and the determination of the $\alpha$ point for that integration. The resulting $(1-\alpha)$ confidence bound in the direction $-\delta$ from $\hat{\beta}$ is then given by

$$\beta = \hat{\beta} - r(\alpha, \delta)s\hat{y}\delta.$$  

5.3 An example

To illustrate the conical test and confidence region method described in 5.1 and 5.2, we consider the special normal model $\hat{y} = X\hat{\beta} + \sigma e$ where $e$ is a sample from a normal $N(0,1)$ distribution and $\sigma$ is unknown. The choice of the special normal distribution in this section devoted
to nonnormal regression provides a simple illustration in which the various steps can be pursued analytically, giving the functional form of \( g_1, g_2, g_3 \), and then the solution to (2.1) and (3.1). It also shows that the conical confidence region for \( \beta \) is identical to the confidence region obtained through the standard F-test procedure.

The joint density of \( s, t \), and \( d \) is obtained from (5.1),

\[
\begin{align*}
f(e) & = k \cdot \exp\left\{-\frac{1}{2} (sXt+svn-pd)^\prime (sXt+svn-pd) \right\} s^{n-1} ds dt dd \\
& = k \cdot \exp\left\{-\frac{1}{2} s^2 t'X'Xt \right\} dt \cdot \exp\left\{-\frac{1}{2} (n-p)s^2 \right\} s^{n-1} ds dd.
\end{align*}
\]

This then gives the conditional distribution of \( s \) and \( t \) given \( d \):

\[
g_1(s, t \mid d) ds dt = k \cdot \exp\left\{-\frac{1}{2} (n-p)s^2 \right\} s^{n-p-1} ds \cdot |X'X|^{1/2} \exp\left\{-\frac{s^2}{2} t'X'Xt \right\} s^p dt,
\]

which has the form of a chi density times a conditional normal. The density of \( t \) conditional on \( d \) is obtained by integrating out \( s \); this standard integration gives a p-variate Student distribution with density.

\[
g_2(t \mid d) = \frac{\Gamma(n/2)}{\pi^{p/2} \Gamma((n-p)/2)} (1 + \frac{t'X'Xt}{n-p})^{-n/2},
\]

that is \( (X'X)^{1/2} t \) has the standard Student(\( n-p \)) distribution in \( \mathbb{R}^p \). The function \( g_2(t) \) attains its maximum at \( t=0 \), so we do not have to relocate \( t \) and can directly compute the OLS given by (2.1); also note \( t \) is independent of \( d \).

To test \( \beta = \beta_0 \), we consider the corresponding value of the pivotal variable \( t_0 = (\beta(y) - \beta_0)/s_y \). For the calculations, we follow the pattern of the alternative
notation in Section 4.2 and let \( v^2 = \frac{t'X'Xt}{t'X'Xt}_0 \) and 
\[ F = \frac{\frac{t'X'Xt}{t'X'Xt}_0}{\frac{v^2}{\frac{t'X'Xt}{t'X'Xt}_0}}. \]
This then gives the OLS
in the direction \( t_0 \):

\[
\int_{1}^{\infty} \left( 1 + v^2 \frac{t'X'Xt}{n-p} \right)^{-n/2} v^{p-1} dv
\]
\[
= \int_{\infty}^{\infty} \left( 1 + v^2 \frac{t'X'Xt}{n-p} \right)^{-n/2} (v^2)^{p/2-1} dv^2
\]
\[
= \int_{0}^{\infty} \left( 1 + \frac{pF}{n-p} \right)^{-n/2} F^{p/2-1} dF
\]
\[
= \int_{F_0}^{\infty} h_{p,n-p}(F) dF
\]

which is the \( F(p,n-p) \) distribution and the OLS is in fact
the ordinary OLS.

Let us now consider the conical \((1-\alpha)\) confidence
region for \( \beta \) around the maximum 'pivotal density' estimate
\( h(y^0) \) of \( \beta \). In the direction \( t_0 \) the parameter \( \beta \)
will not be acceptable if the corresponding OLS \( \leq \alpha \), that
is
\[ \int_{F_0}^{\infty} h_{p,n-p}(F) dF \leq \alpha \]

where the \( t_0 \), \( F_0 \) use the value \( \beta \) being tested. Thus the (1-\( \alpha \)) conical confidence region for \( \tilde{\beta} \) is the set of \( \tilde{\beta} \) such that \( F_0 \leq F_{\alpha} \). This coincides with the standard (1-\( \alpha \)) confidence region for \( \tilde{\beta} \) for this normal regression model \( y = X\tilde{\beta} + \tilde{\sigma}e \) when \( \tilde{\sigma} \) is unknown.

5.4 Discussion

An overall (1-\( \alpha \)) confidence region obtained by whatever theory and method would reasonably need plotting or computer plotting unless it is readily available in a linear or quadratic type formula as with the normal. The conical procedure allows the plotting of the (1-\( \alpha \)) boundary point in any chosen direction on the parameter space by a subroutine call (involving a one-dimensional integration). With adequate integration routines this calculation can be fast and accurate with a mainframe capacity, and thus permit interactive graphical displays of the confidence region.

Any pair of parameter coordinates can have the boundary points of the (1-\( \alpha \)) region displayed on a 2-dimensional graph and anomalous shape or contour formation explored; this would be done for selected values of the other coordinates. By rotation movement techniques, the same can be extended to the examination of three parameter coordinates. In the same manner any orthogonal combination of two or three coordinates can be examined for various values of remaining orthogonal combinations.

In some contexts, a confidence interval for a particular coordinate may be wanted. If the graphical exploration
indicates that the regions are approximately elliptical then a first order approximation can be obtained by using a normal-to-chi tail-type adjustment (the elliptical drop off plus the central limit theorem for likelihood effects can give this approximate normal likelihood/pivotal shape); let $z_{\alpha} = \chi_{\alpha^2}$. Then determine $\alpha^*$ by the normal to chi-square(p) relation and calculate the $(1-\alpha^*)$ conical confidence bound in the positive and negative directions from $\hat{\beta}$. Various transect methods will be discussed elsewhere.

6. Conclusion
Conical tests provide computer access to multiparameter tests and confidence regions. The methods have been described in terms of pivotal equations for a standard model or in terms of error for an error model. The same conical procedures will give a $(1-\alpha)$ Bayesian region by conical integration from the maximum posterior probability estimate.

References
[1] Pick, G.H., Computer implementation program for location-scale analysis, Department of Mathematics, University of Toronto, 1976.


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